DAILY UPDATE FOR MATH 147 SPRING 2021

Monday, February 1. We began class by introducing ourselves, giving a brief discussion of our majors and interests. We then went over the syllabus for the course, noting course procedures, dates for exams and grading schemes. This was followed by an overview of the topics to be covered in the course. We started by noting that the notion of limit might be described as the key component of calculus (of any ilk) that distinguishes it from the mathematics students may have previously encountered, and thus, still plays a central role in Calculus III. After reminding ourselves how the notion of limit is used to define the derivative of a function of one variable, we talked briefly about how this might lead to multiple derivatives for a function of more than one variable, and that extreme values for a function of two variables might be determined by a *tangent plane* to the graph of that function. We then gave a brief overview of how the integration process works in general: One always has a function to integrate (the *integrand*) and a domain of integration. We then described how the integration process works in all scenarios we will encounter during the semester. Namely, starting with a domain of integration and a function defined on that domain, we proceed as follows:

(i) Subdivide the domain of integration into small portions of a similar type, e.g, if the domain of integration is a solid, subdivide into smaller solids; if the domain of integration is a curve, subdivide into smaller curves.
(ii) Choose a point in each subdivision and evaluate the function at that point.
(iii) Multiply the answer in (ii) by the size of the subdivision, e.g., volume if a solid, length if a curve.
(iv) Add the quantities in (iii).
(v) Take the limit of the sums in (iv) as the size of the subdivisions tend to zero.

The resulting numerical value depends only on the function and the underlying geometry of the domain of integration. We noted that the real challenge is to calculate this quantity. We finished by surmising that if we use a curve in $\mathbb{R}^3$ to model a wire, then integrating a temperature function along the curve could be used to yield an average temperature along the wire (after dividing by the length of the wire), and that if we model a fluid flowing across the boundary of a container using a vector field, then integrating that vector field along a surface should give us the *flux* of the fluid across the boundary of the container. The point of our course is to study all of these notions in detail.

Tuesday, February 2. We began our discussion of functions of several variables, starting with the examples $f(x,y) = x^2 + y^2$, $g(x,y,z) = \frac{1}{x^2+y^2+z^2}$, and $h(x_1,x_2,\ldots,x_n) = x_1 + x_2 + \cdots + x_n$. We noted that $f(-2,3) = (-2)^2 + 3^2 = 13$, $g(1,2,3) = \frac{1}{1^2+2^2+3^2} = \frac{1}{14}$, and $h(1,2,\ldots,n) = 1+2+\cdots+n = \frac{n(n+1)}{2}$.

We then noted that the range and domain for functions of several variables, have the same meaning as for functions of a single variable, namely the domain is the set of allowable inputs and the range is the set of possible outputs. In the case of a function of two variables, say, the allowable inputs will elements of $\mathbb{R}^2$, but the the outputs are always real numbers. For the functions defined above, we have:

(i) $f(x,y)$: Domain $= \mathbb{R}^2$ and range $= \{\alpha \in \mathbb{R} \mid \alpha \geq 0\}$.
(ii) $g(x,y,z)$: Domain $\mathbb{R}^3 \backslash (0,0,0)$ and range $= \{\alpha \in \mathbb{R} \mid \alpha > 0\}$.
(iii) $h(x_1,x_2,\ldots,x_n)$: Domain $= \mathbb{R}^n$ and range $= \mathbb{R}$.

We then noted that for the function $t(x,y) = \frac{1}{\sqrt{1-x^2-y^2}}$, the domain is the interior of the unit circle of radius one centered at the origin in $\mathbb{R}^2$ and the range is $\{\alpha \in \mathbb{R} \mid \alpha \geq 1\}$. After this we discussed graphing functions of two variables $z = f(x,y)$ and how the level curves $f(x,y) = c$ for different values of c help to understand the graph of $f(x,y)$. We also cautioned that the level curves do not give a complete picture of the graph, since for example, the level curves of $f(x,y) = x^2 + y^2$ and $g(x,y) = \sqrt{x^2+y^2}$ are circles of increasing radii, as $c > 0$ increases, but the graph of $f(x,y)$ is a *paraboloid* (see here) while the graph of $g(x,y)$ is a *cone* (see here). We noted that the difference between these two surfaces can be seen by taking curves obtained by setting $x = c$, and in particular $x = 0$. In this cross section, the graph of $f(x,y)$ is a parabola while the graph of $g(x, y)$ is the graph of $z = |y|$ in the yz-plane, i.e., two rays emanating from the origin at 45 degrees. We then sketched the graph of $z = \frac{1}{x^2+y^2}$, noting that it looks like an ever expanding inverted funnel whose increasing base asymptotically approaches the xy-plane and whose narrowing neck tends to positive infinity, (see here).

We ended class by briefly defining level surfaces, and noted that the sphere $x^2 + y^2 + z^2 = r^2$ of radius r centered at the origin is an example of a level surface in $\mathbb{R}^3$.

Wednesday, February 3. We continued with the discussion of level surfaces, by defining a level surface to be the set of points in $\mathbb{R}^3$ satisfying an equation of the form $f(x, y, z) = c$, with c a constant. We noted that the graph a function of two variables can be interpreted as a level surface, since if $z = g(x, y)$, and we set $f(x, y, z) = z - g(x, y)$, then the graph of $z - g(x, y)$ is the same as the level surface $f(x, y, z) = 0$. When then looked at pictures of graphs the six basic *quadric surfaces*: elliptic paraboloid, hyperbolic paraboloid, ellipsoid, double cone, hyperboloid of one sheet, and hyperboloid of two sheets, (see here).

We next recalled the definition of the distance between points in $\mathbb{R}^2$, $\mathbb{R}^2$ or $\mathbb{R}^n$, recalling for example that the distance between (a, b) and (c, d) in $\mathbb{R}^2$ is equal to $\sqrt{(a-c)^2 + (b-d)^2}$, while in general, the distance from $(x_1, x_2, \ldots x_n)$ to $(y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ equals $\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2}$. Following the notation in our text, we wrote $|(a, b) - (c, d)|$ for the distance between (a, b) and (c, d).

We then discussed the concept of limit, for a function of one variable. Just as $\lim_{x\to a} f(x) = L$ intuitively mean that the values of $f(x)$ approach L as x approaches x, it should be the case that $\lim_{(x,y)\to(a,b)} f(x, y) = L$ means that the values of $f(x, y)$ approach the real number L, as (x, y) approaches (a, b). We noted that while limits for functions of one variable involve a approaching x from either the right or left, for limits with functions of two variables, there are infinitely many ways (x, y) can approach (a, b). On the other hand, using the notion of distance, the definition of a limit still takes a very similar form as in one variable: Namely, we say $\lim_{(x,y)\to(a,b)} f(x, y) = L$ if for every $\epsilon > 0$, there exists $\delta > 0$, such that $|(x, y) - (a, b)| < \delta$ implies $|x - L| < \epsilon$.

Before calculating some examples, we noted that the following rules for taking limits hold, and that these are the same rulles that apply when taking limits of function of one variable.

Theorem 12.2.1 Basic Limit Properties of Functions of Two Variables

Let b, x_0, y_0, L and K be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x, y) = K.$$

The following limits hold.

1. Constants: $\lim_{(x,y)\to(x_0,y_0)} b = b$
2. Identity $\lim_{(x,y)\to(x_0,y_0)} x = x_0$; $\lim_{(x,y)\to(x_0,y_0)} y = y_0$
3. Sums/Differences: $\lim_{(x,y)\to(x_0,y_0)} \big(f(x, y) \pm g(x, y)\big) = L \pm K$
4. Scalar Multiples: $\lim_{(x,y)\to(x_0,y_0)} b \cdot f(x, y) = bL$
5. Products: $\lim_{(x,y)\to(x_0,y_0)} f(x, y) \cdot g(x, y) = LK$
6. Quotients: $\lim_{(x,y)\to(x_0,y_0)} f(x, y)/g(x, y) = L/K$, $(K \neq 0)$
7. Powers: $\lim_{(x,y)\to(x_0,y_0)} f(x, y)^n = L^n$

Thus, for example,

$$\begin{aligned}\lim_{(x,y)\to(1,2)} \frac{3x^2y+4xy}{x+y} &= \frac{\lim_{(x,y)\to(1,2)} 3x^2+4xy}{\lim_{(x,y)\to(1,2)} x+y}\\ &= \frac{3\lim_{(x,y)\to(1,2)} x^2 + 4\lim_{(x,y)\to(1,2)} xy}{\lim_{(x,y)\to(1,2)} x + \lim_{)x,y)\to(1,2)} y}\\ &= \frac{3(\lim_{x\to 1} x)^2 + 4(\lim_{x\to 1} x)(\lim_{y\to 2} y)}{\lim_{x\to 1} x + \lim_{y\to 2} y}\\ &= \frac{3\cdot 1^2 + 4\cdot 1\cdot 2}{1+3}\\ &= \frac{11}{3}.\end{aligned}$$

We also noted that when applying the rules above to $\lim_{(x,y)\to(a,b)} f(x,y)$ it is crucial to use limits, and not merely substitute (a,b) for (x,y). For example, let

$$f(x) = \begin{cases} x, & \text{if } x \neq 1\\ 2, & \text{if } x = 1,\end{cases}$$

so that $\lim_{x\to 1} f(x) = 1 \neq f(1)$. Thus, $\lim_{(x,y)\to(1,2)} f(x)\cdot(x^2+3xy) = 1\cdot(1^2+3\cdot 1\cdot 2) = 7$.

Thursday, February 4. Before continuing our discussion of limits involving functions of two variable, we did a thought experiment with level surfaces. I asked the class to imagine a two-dimensional world (as in the book *Flatland*) and our goal was to convince the inhabitants of the planar world of the existence of our three-dimensional world. The idea is to describe what those occupants would see if we first held a sphere of radius one above the plane in which they live, and then slowly pass our sphere through their plane. After discussing this, and noting that the planar world would first see a point emerge from nowhere, and then the level curves of our sphere, and finally a disappearing point (the north point of our sphere), I asked how someone in a four-dimensional world could similarly convince us of the existence of a sphere in Euclidean four-space. We agreed that, as before, we would see a point emerge from nowhere, only to grow into increasing spheres, then decreasing spheres - the level surfaces - and finally a disappearing point. We noted that analytically we were seeing the points on the spheres $x^2+y^2+z^2+w^2=r^2$, with $0\leq r\leq 1$ and $w=0$.

We then continued our discussion of the limits involving functions of two variables. We did two easy examples using the limit properties from the previous lecture, where the limit could be evaluated essentially by substitution. We also showed how very special limits like $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$ could be evaluated using polar coordinates. We then did some examples where the limit was shown not to exist by approaching the target in question along different paths. For example, in considering $\lim_{(x,y)\to(0,o)} \frac{2xy}{x^2+y^2}$, we noted that taking the limit along the line $y=0$ gives 0, while taking the limit along the line $y=x$ yields 1, showing that the full limit does not exist. We also noted that having a uniform limit as (x,y) approaches $(0,0)$ along every straight line does not imply that the limit exists, as the example $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4}$ shows. We also did a couple of limits like $\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$, whose limit exists, even though the limits of both numerator and denominator are 0.

We finished defining the function $f(x,y)$ to be continuous at (a,b) if and only if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Friday, February 5. We began with two examples showing how to verify a limit using the ϵ, δ definition. We noted that 'working backwards' to find δ given ϵ can often be helpful. For the example $\lim_{(x,y)\to(a,b} x = a$, we noted that given $\epsilon > 0$, we can take $\delta = \epsilon$, so that $|(x,y)-(a,b)| < \epsilon$ implies $|x-a| < \epsilon$. For the function $f(x,y) = \frac{y^4}{x^2+y^2}$, we had shown previously using trig substitution that $\lim_{(x,y)\to(0,0)} \frac{y^4}{x^2+y^2} = 0$. We then saw that taking $\delta = \sqrt{\epsilon}$ was sufficient for $|(x,y)-(0,0)| < \delta$ to imply $|\frac{y^4}{x^2+y^2} - 0| < \epsilon$, thereby verifying the limit using the limit definition.

We then recalled that $f(x,y)$ is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. We noted that the limit properties discussed in Wednesday's lecture imply analogous properties for continuity, i.e., the functions x and y are continuous at (a,b) for all $(a,b) \in \mathbb{R}^2$, and if $f(x,y)$, $g(x,y)$ are continuous at (a,b) then so are sums, differences, products, quotients (if the denominator is not zero), and powers of $f(x,y), g(x,y)$. Moreover, if $h(t)$ is defined and continuous on the range of $f(x,y)$, then $h(f(x,y))$ is continuous at (a,b) whenever $f(x,y)$ is continuous at (a,b). We then split the class into three breakout groups where each group discussed and worked on the following problems:

1. Calculate $\lim_{(x,y)\to(3,4)} \frac{x^2+2xy}{xy^2}$. Justify your answer.

2. If $f(x) = \begin{cases} \cos(x), & \text{if } x \neq 0 \\ 5, & \text{if } x = 0, \end{cases}$ and $g(y) = \begin{cases} \sqrt{y}, & \text{if } y \neq 4 \\ \pi, & \text{if } y = 4, \end{cases}$, find $\lim_{(x,y)\to(0,4)} f(x)g(y)$.

3. Find $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$.

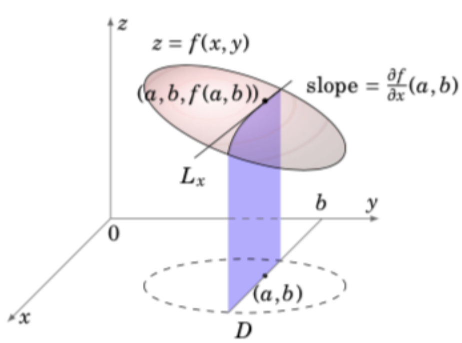
4. Analyze the $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$ by first taking the limit along any line $y = mx$ lying in the domain of $\frac{\sin(xy)}{x+y}$ and then taking the limit along the path $y = -\sin(x)$.

The class was brought back together and solutions to the problem were discussed.

Monday, February 8. We began our discussion of partial derivatives. We noted that given a function $f(x,y)$, if we start at the point (a,b), then there are infinitely many directions moving away from (a,b) for which we could seek the rate of $f(x,y)$. Our analysis began with the rate of change of $f(x,y)$ at (a,b) in a direction parallel to the x-axis. We noted that this can be analyzed by intersecting the graph of $z = f(x,y)$ with the plane $y = b$. We observed that doing so reduces the problem to calculating the derivative of the function $c(x) := f(x,b)$ at $x = a$. Thus,

$$c'(a) = \lim_{h\to 0} \frac{c(a+h) - c(a)}{h} = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$

If this limit exists, this is the *partial derivative* of $f(x,y)$ with respect to x at (a,b), which we denote by $\frac{\partial f}{\partial x}(a,b)$ or $f_x(a,b)$. Geometrically, we can think of $\frac{\partial f}{\partial x}(a,b)$ as the slope of the line passing through the point $(a,b,f(a,b))$ tangent to the curve $z = f(x,b)$ lying on the graph of $z = f(x,y)$.



(a) Tangent line L_x in the plane $y = b$

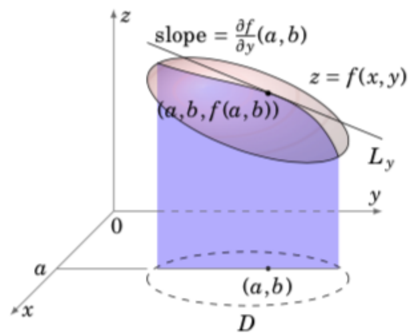
We then used the limit definition to calculate $f_x(1,1)$ for the function $f(x,y) = 2x+3y$ and $g_x(2,-1)$ for the function $g(x,y) = x^3y + 3xy^2$. The resulting values were 2 and -9. We then repeated the limit calculations to find the general values of $f_x(x,y)$ and $g_x(x,y)$, and observed that $f_x(x,y) = 2$ and $g_x(x,y) = 3x^2y + 3y^2$. This suggests that in general, when the relevant limits exist, the partial derivative of an arbitrary $f(x,y)$ with respect to x is obtained by differentiating the expression for $f(x,y)$ with respect to x, treating y as a constant. We then did this direct calculation for $h(x,y) = 6x^2y^3e^{x^2+2y^2} + 5\cos(xy)$, obtaining

$$\frac{\partial h}{\partial x} = 12xy^3e^{x^2+2y^2} + 6x^2y^3e^{x^2+2y^2}(2x) - 5\sin(xy)\cdot y.$$

We then noted that the discussion above applies equally well to the variable y, so that one defines the partial derivative of $f(x,y)$ with respect to y at (a,b) as

$$\frac{\partial f}{\partial y}(a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h},$$

provided the limits exists. This then represents the slope of the line tangent tangent to the graph $z = f(x,y)$ at (a,b) along the curve obtained by intersecting the graph of $z = f(x,y)$ with the plane $x = a$.



(b) Tangent line L_y in the plane $x = a$

Accordingly, the function $\frac{\partial f}{\partial y}$ is calculated by differentiating $f(x,y)$ with respect to y, treating x as a constant. Doing so for $h(x,y)$ above yields,

$$\frac{\partial h}{\partial y} = 18x^2y^2e^{x^2+2y^2} + 6x^2y^3e^{x^2+2y^2}(4y) - 5\sin(xy)\cdot x.$$

We concluded the lecture by recalling how to describe a line in $\mathbb{R}^3$. If $P = (x_0, y_0, z_0) \in \mathbb{R}^3$ is a point and $\vec{\mathrm{D}} = (d_1, d_2, d_3)$ is a direction vector, then the line through P in the direction of $\vec{\mathrm{D}}$ is given parametrically by:

$$L(t) = P + t\cdot\vec{\mathrm{D}} = (x_0 + td_1, y_0 + td_2, z_0 + td_3).$$

In a future lecture, we will see that from our discussion above, we will be able to describe tangent lines to the graph of $z = f(x,y)$ in the directions given by the x and y axes.

Tuesday, February 9. We began class with Quiz 1. We then reviewed the definitions of the partial derivatives of $f(x,y)$ and how to calculate them. We noted that the definition easily carries over to functions of several variables, the general case being a function $f(x_1, x_2, \ldots, x_n)$. In this case, we defined the partial derivative of $f(x_1,\ldots,x_n)$ with respect to x_i at $(a_1,\ldots,a_n)$ to be

$$\frac{\partial f}{\partial x_i}(a_1,a_2,\ldots,a_n) = \lim_{h\to 0}\frac{f(a_1,\ldots,a_{i-1},a_i+h,a_{i+1},\ldots,a_n) - f(a_1,a_2,\ldots,a_n)}{h},$$

if the limit exists. We then noted that to calculate partial derivatives of functions of many variables with respect to a given variable, we differentiate with respect to that variable, treating all remaining variables as constants. We then calculated some of the partial derivatives of the function

$$h(x,y,z,u,w) = x^3y^2z^5 + 3e^{xu^2w^3} + \cos(u^2+w^2+x^2),$$

obtaining:

(i) $\frac{\partial h}{\partial x} = 3x^2y^2z^5 + 3e^{xu^2w^3}\cdot(u^2w^3) + 0.$

(ii) $\frac{\partial h}{\partial w} = 0 + 3e^{xu^2w^3}\cdot(3xu^2w^2) - \sin(u^2+w^2+z^2)\cdot 2w.$

(iii) $\frac{\partial h}{\partial z} = 5x^3y^2z^4 + 0 - \sin(u^2+w^2+z^2)\cdot 2z.$

We followed this by noting that many of the usual properties of the derivative carry over for partials, including rules for differentiating sums, products and quotients. We noted that the chain rule for partials is more complicated. For a composition of the type $h(f(x,y))$, where $h(t)$, say, is a function of one variable, the chain rule works as expected, e.g., $\frac{\partial h(f(x,y))}{\partial x} = h'(f(x,y))\cdot\frac{\partial f}{\partial x}$. However, if, for example, $x = x(u,v)$,

$y = y(u, v)$ are functions of u and v, so that $f(x, y)$ is in turn a function of u and v, then the formulas for $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ require further discussion at a later date.

We ended class by discussing second order partial derivatives, that are defined as follows:

1. The **second partial derivative of f with respect to x then x** is

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = (f_x)_x = f_{xx}$$

2. The **second partial derivative of f with respect to x then y** is

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = (f_x)_y = f_{xy}$$

Similar definitions hold for $\frac{\partial^2 f}{\partial y^2} = f_{yy}$ and $\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$.

The second partial derivatives f_{xy} and f_{yx} are **mixed partial derivatives**.

We stated that we will study a theorem of fundamental importance that guarantees the equality of the mixed partials f_{xy} and f_{yx}, but noted that for most familiar functions, this equality does hold. We then verified this property for the function $f(x, y) = x^3y^2 + 7xy^5 + e^{xy}$.

Wednesday, February 10. The purpose of today's lecture was to derive the parametric equations for lines tangent to the graph of $z = f(x, y)$ in the x and y directions, and to use this information to find the algebraic equation of the tangent plane containing those lines. Towards those ends, we began by reviewing how to find the parametric equation of a line in $\mathbb{R}^3$ passing through a point P in the direction of the vector $\vec{\mathrm{D}}$, obtaining $L(t) = P + t \cdot \vec{\mathrm{D}}$, as in Monday's lecture. We recalled that a plane in $\mathbb{R}^3$ is determined by a point $P = (a, b, c)$ and a normal vector $\vec{n} = (n_1, n_2, n_3)$ and the equation of that plane is given by $n_1(x - a) + n_2(y - b) + n_3(z - c) = 0$.

We then revisited the argument from Monday that lead to the fact that $\frac{\partial f}{\partial x}(a, b)$ gives the slope of the line tangent in the x direction to the graph of $z = f(x, y)$ at $(a, b, f(a, b))$. We noted that in the plane $y = b$, the equation of the tangent line is $z = \frac{\partial f}{\partial x}(a, b)(x - a) + f(a, b)$. Replacing x by the parameter t, and considering this equation as determining the z coordinate of the tangent line in $\mathbb{R}^3$, we obtained that the line in question is given parametrically by

$$L_x(t) = (t, b, \frac{\partial f}{\partial x}(a, b)(t - a) + f(a, b)) = (a, b, f(a, b)) + (t - a)(1, 0, \frac{\partial f}{\partial x}(a, b)).$$

Adjusting the parameter t, so that the line passes though $(a, b, f(a, b))$ when $t = 0$, gives

$$L_x(t) = (a, b, f(a, b)) + t(1, 0, \frac{\partial f}{\partial x}(a, b)).$$

We then calculated the tangent line $L_x(t)$ for $f(x, y) = x^3y^2 + 2xy + 7$ and the point $(-1, 2, f(-1, 2))$. Since $f(-1, 2) = -1$ and $\frac{\partial f}{\partial x}(-1, 2) = 16$, we obtained

$$L_x(t) = (-1, 2, -1) + t(1, 0, 16).$$

We then noted that, in general, the tangent line in the y direction to the graph of $z = f(x, y)$ at $(a, b, f(a, b))$ is given by

$$L_y(t) = (a, b, f(a, b)) + t(0, 1, \frac{\partial f}{\partial y}(a, b)).$$

We calculated $L_y(t)$ for the example above, obtaining,

$$L_y(t) = (-1, 2, -1) + t(0, 1, -6).$$

We then noted that the general analysis above yields two tangent vectors to the graph of $z = f(x,y)$ at $(a,b,f(a,b))$, namely $(1,0\frac{\partial f}{\partial x}(a,b))$ and $(0,1,\frac{\partial f}{\partial y}(a,b))$. Since the cross product of these vectors gives a normal vector to the tangent plane, we calculated

$$(1,0\frac{\partial f}{\partial x}(a,b)) \times (0,1,\frac{\partial f}{\partial y}(a,b)) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial f}{\partial x}(a,b) \\ 0 & 1 & \frac{\partial f}{\partial y}(a,b) \end{vmatrix} = -\frac{\partial f}{\partial x}(a,b)\vec{i} - \frac{\partial f}{\partial y}(a,b)\vec{j} + \vec{k}.$$

Thus an equation for the plane tangent to the graph of $z = f(x,y)$ at $(a,b,f(a,b))$ is given by

$$-\frac{\partial f}{\partial x}(a,b)(x-a) - \frac{\partial f}{\partial y}(a,b)(y-b) + 1 \cdot (z - f(a,b)) = 0.$$

Re-writing, we obtained the more standard form for the tangent plane in question

$$z = \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) + f(a,b).$$

Applying this formula to the example above, we saw that the tangent plane to the graph of $f(x,y) = x^3y^2 + 2xy + 7$ at (1-,2,-1) is given by

$$z = 16(x+1) - 6(y-1) - 1.$$

Thursday, February 11. We began class by reviewing the equations of tangent lines and tangents planes derived in Wednesday's lecture. We noted that we cannot always expect a tangennt plane to exist at every point of the graph of $z = f(x,y)$, just like a tangent line to the graph of $y = f(x)$ need not exist at every point. We mentioned that we will discuss this at greater length next week.

We then defined the directional derivative of the function $f(x,y)$ at (a,b) in the direction of the unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j}$:

$$D_{\vec{u}}f(a,b) = \lim_{h\to 0} \frac{f(a+hu_1, b+u_2h) - f(a,b)}{h},$$

noting that this is the rate of change of $f(x,y)$ at (a,b) along the line $(a,b) + t\cdot\vec{u}$, assuming the limit exists. We emphasized the importance of taking a unit vector in this definition, so that the quantity calculated only depends upon the function f and the direction of the direction vector, and not also on the magnitude of the direction vector.

We then used this definition to calculate $D_{\vec{u}}f(1,2)$ for $f(x,y) = x^2y$ in the direction of $\vec{u} = \frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{j}$, obtaining $2\sqrt{2} + \frac{\sqrt{2}}{2}$. We then redid this limit for an arbitrary point (x,y), getting:

$$\begin{aligned} D_{\vec{u}}f(x,y) &= \lim_{h\to 0} \frac{f(x+hu_1, y+hu_2) - f(x,y)}{h} \\ &= \lim_{h\to 0} \frac{(x+h\frac{\sqrt{2}}{2})^2(y+h\frac{\sqrt{2}}{2}) - x^2y}{h} \\ &= \lim_{h\to 0} \frac{x^2y + xyh\sqrt{2} + h^2\cdot\frac{y}{2} + x^2h\frac{\sqrt{2}}{2} + xh^2 + \frac{\sqrt{2}}{4}h^3 - x^2y}{h} \\ &= \lim_{h\to 0} xy\sqrt{2} + h\cdot\frac{y}{2} + x^2\frac{\sqrt{2}}{2} + xh + \frac{\sqrt{2}}{4}h^2 \\ &= \sqrt{2}xy + \frac{\sqrt{2}}{2}x^2. \end{aligned}$$

We observed that this last expression is $(2xy\vec{i} + x^2\vec{j})\cdot\vec{u} = (\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j})\cdot\vec{u}$. Upon the basis of this observation, we noted (without justification), that in general, one can calculate the directional derivative as

$$D_{\vec{u}}f(a,b) = (\frac{\partial f}{\partial x}(a,b) + \frac{\partial f}{\partial y}(a,b)\vec{j})\cdot\vec{u}.$$

For $f(x,y) = x^2y$ and the point (1,2), $\frac{\partial f}{\partial x}(1,2) = 4$, $\frac{\partial f}{\partial y}(1,2) = 1$, so that

$$D_{\vec{u}}f(1,2) = (4\vec{i} + \vec{j}) \cdot (\frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{j}) = 2\sqrt{2} + \frac{\sqrt{2}}{2},$$

which is what we obtained via the limit definition.
This lead to the definition of the *gradient*, ∇f, of a scalar function f:

(i) For $f(x,y)$, $\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$.
(iii) For $f(x,y,z)$, $\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$.
(iii) For $f(x_1, x_2, \ldots, x_n)$, $\nabla f = \frac{\partial f}{\partial x_1}\vec{e_1} + \frac{\partial f}{\partial x_2}\vec{e_2} + \cdots + \frac{\partial f}{\partial x_n}\vec{e_n}$, where the vector $\vec{e_i}$ is the vector in $\mathbb{R}^n$ all of whose coordinates are zero, except the ith coordinate, which is 1.

Using this notation, then in the cases above we have

(i) $D_{\vec{u}}f(a,b) = \nabla f(a,b) \cdot \vec{u}$.
(ii) $Duf(a,b,c) = \nabla f(a,b,c) \cdot \vec{u}$.
(iii) $D_{\vec{u}}f(a_1, a_2, \ldots, a_n) = \nabla f(a_1, a_2, \ldots, a_n) \cdot \vec{u}$.

where in each case $\vec{u}$ is an appropriate unit vector.

We then mentioned that ∇ is a *differential operator* that turns scalar value functions into vector valued function through the differentiation process. As such, one can expect ∇ to have similar properties that hold upon differentiation. Indeed, the following properties hold:

(i) $\nabla(f+g) = \nabla f + \nabla g$.
(ii) $\nabla(cf) = c\nabla f$, for the constant c.
(iii) $\nabla(fg) = f\nabla g + g\nabla f$.
(iv) For $h(t)$, $\nabla h(f) = h'(f)\nabla f$.

We ended class by using the gradient to calculate the directional derivative of $f = x^2y^3 + 4xz^2 + 17$, at (1,-1,1) in the direction of the vector $= 2\vec{i} + 3\vec{j} + 4\vec{k}$. We first noted we need to turn $\vec{v}$ into a unit vector: $\vec{u} = \frac{1}{\sqrt{29}}\vec{v}$. Moreover, we have $\nabla f = (2xy^3 + 4z^2)\vec{i} + 3x^2y^2\vec{j} + 8xz\vec{k}$, so that $\nabla f(1,-1,1) = 2\vec{i} + 3\vec{j} + 8\vec{k}$. Thus, the required directional derivative is:

$$\begin{aligned} D_{\vec{u}}f(1,-1,1) &= \nabla f(1,-1,1) \cdot \vec{u} \\ &= (2\vec{i} + 3\vec{j} + 8\vec{k}) \cdot \frac{1}{\sqrt{29}}(2\vec{i} + 3\vec{j} + 4\vec{k}) \\ &= \frac{1}{\sqrt{29}}(4 + 9 + 32) \\ &= \frac{45}{\sqrt{29}}. \end{aligned}$$

Friday, February 12. We began by reviewing the definition of the directional derivative and the gradient. We then showed that by assuming a special case of the chain rule, we could derive the formula $D_{\vec{u}}f(a,b,c) = \nabla f(a,b,c) \cdot \vec{u}$. The version of the chain rule we used is:

$$\frac{df(x(t), y(t), z(t))}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}.$$

If we write $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$, and set $g(t) := f(a + tu_1, b + tu_2, c + tu_3)$, then

$$\begin{aligned} g'(0) &= \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} \\ &= \lim_{h \to 0} \frac{f(a + (0+h)u_1, b + (0+h)u_2, c + (0+h)u_3)) - f(a,b,c)}{h} \\ &= \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2, c + hu_3) - f(a,b,c)}{h} \\ &= D_{\vec{u}}f(a,b,c). \end{aligned}$$

We then noted that by the chain rule, $g'(0) = \frac{\partial f}{\partial x}(a,b,c)\cdot u_1 + \frac{\partial f}{\partial y}(a,b,c)\cdot u_2 + \frac{\partial f}{\partial z}(a,b,c)\cdot u_3 = \nabla f(a,b,c)\cdot \vec{u}$. We concluded that this gives the equality $D_{\vec{u}}f(a,b,c) = \nabla f(a,b,c)\cdot\vec{u}$.

We then noted that since $\nabla f(a,b,c)\cdot\vec{u} = |\nabla f(a,b,c)|\cdot|\vec{u}|\cos(\theta) = |\nabla f(a,b,c)|\cos(\theta)$, where θ is the angle between $\nabla f(a,b,c)$ and $\vec{u}$, $D_{\vec{u}}f(a,b,c)$ achieves its greatest value when $\vec{u}$ points in the same direction as $\nabla f(a,b,c)$, and moreover, the rate of change in that direction is $|\nabla f(a,b,c)|$. Likewise, we noted that $-\nabla f(a,b,c)$ points in the direction in which the rate of change at (a,b,c) is the least, and that rate of change is $-|\nabla ff(a,b,c)|$.

We then moved to breakout rooms to work on the following Practice Problems:

1. Calculate all first order partial derivatives for $f(x,y,z,w) = x^2y^3zw^5 + e^{x^2+y^3+z^2w^5}$. Then verify that $f_{zw} = f_{wz}$.

2. For $f(x,y) = x^2 + 4xy^2 + y^3$, find the tangent lines $L_x(t)$ and $L_y(t)$ at $(2,-1,f(2,-1))$. Find the tangent plane to the graph of $f(x,y)$ at the point $(2,-1,f(2,-1))$.

3. For the function $f(x,y,z) = xyz$:
   - (i) Use the limit definition to calculate the directional derivative of $f(x,y,z)$ at $(1,2,-1)$ in the direction of the vector $\vec{v} = \vec{i} + \vec{j} + 2\vec{k}$.
   - (ii) Verify your answer in (i) by using the formula $D_{\vec{u}}f(a,b,c) = \nabla f(a,b,c)\cdot\vec{u}$.
   - (iii) Now find the directional derivative of $f(x,y,z)$ at (1,2,-1) in the direction of $\nabla f(1,2,-1)$. How does your answer compare to the one in (ii)?

4. For $f(x,y) = x^2 + 4xy^2 + y^3$, find the parametric equation of the line tangent to the graph of $f(x,y)$ at $(2,-1,f(2,-1))$ in the direction of $\vec{u} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$.

5. Verify the property $\nabla(fg) = f\nabla g + g\nabla f$.

We reconvened with five minutes left to discuss how to find the tangent line to the graph of $z = f(x,y)$ at (a,b) in the direction of a unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j}$. Knowing this, makes problem 4 an easy calculation. We noted that since $\frac{\partial f}{\partial x}(a,b) = D_{\vec{i}}f(a,b)$, and $\frac{\partial f}{\partial y}(a,b) = D_{\vec{j}}f(a,b)$, we can employ the same reasoning that lead to the tangent lines $L_x(t)$ and $L_y(t)$. The argument takes place in the the vertical plane whose intersection with the xy-plane is the line $(a,b) + t\vec{u}$. This plane intersects the graph of $f(x,y)$ in the curve $z = f(a+tu_1, b+tu_2)$, and we noted that the argument above shows that the slope of the tangent line at $(a,b,f(a,b)$ (i.e., when $t=0$) is $D_{\vec{u}}f(a,b)$. Thus, in this plane, the tangent line is $z = D_{\vec{u}}f(a,b)t + f(a,b)$. Since in this plane $x = a + u_1t$ and $y = b + u_2t$, we see that in $\mathbb{R}^3$, the full tangent line is given by

$$(a+tu_1, b+tu_2, D_{\vec{u}}f(a,b)t + f(a,b)) = (a,b,f(a,b,)) + t(u_1,u_2,D_{\vec{u}}f(a,b)),$$

which also shows that $(u_1,u_2,D_{\vec{u}}f(a,b))$ is a tangent vector to the graph of $z = f(x,y)$ at $(a,b,f(a,b))$ in the direction of $\vec{u}$.

Monday, February 15. We began class by discussing the following theorem.

Theorem (Equality of mixed partials). Given $f(x,y)$ and a point $(a,b)\in\mathbb{R}^2$, suppose that there is an open disk D about (a,b) such that f_{xy} and f_{yx} exist and are continuous at every point in D. Then

$$f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0),$$

for all $(x_0,y_0)\in D$.

We then worked through the following example, which illustrates the need for the continuity assumptions in the theorem.

Example. Suppose $f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y) \neq 0 \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$. We showed:

(i) $f(x,y)$ is continuous at (0,0), noting that it is also continuous for $(x,y) \neq (0,0)$. For this, we worked out the details of the bonus problem from Week 1.
(ii) $f_x(x,y)$ and $f_y(x,y)$ exist and are continuous everywhere, including at (0,0).
(ii) $f_{xy}(x,y)$ and $f_{yx}(x,y)$ exist everywhere, and that $f_{xy}(0,0) = -1$, while $f_{yx}(0,0) = 1$.
(iv) $f_{xy}(x,y)$ is not continuous at (0,0) by showing that $\lim_{(x,y)\to(0,0)} f_{xy}(x,y)$ does not exist.

Thus, in light of the theorem, (iv) above explains why, for this example, $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Tuesday, February 16. All KU classes were cancelled.

Wednesday, February 17. We began class with Quiz 2. We then began a discussion of what should constitute a good linear approximation to $f(x,y)$ at (a,b). Geometrically, this should mean that there is a well defined tangent plane to the graph of $z = f(x,y)$ at $(a,b,f(a,b))$. In other words, does the proposed tangent plane $z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$ do a good job of approximating $f(x,y)$ in the vicinity of (a,b)?

We reviewed the situation for the one variable case. Suppose $f'(a)$ exists and we set $L(x) = h'(a)(x-a)+f(b)$, the tangent line to the graph of $y = f(x)$ at $x = a$. Since$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$, we showed

$$\begin{aligned} \lim_{x\to a} \frac{f(x) - L(x)}{x-a} &= \lim_{x\to a} \frac{f(x) - \{f'(a)(x-a) + f(a)\}}{x-a} \\ &= \lim_{x\to a} \frac{f(x) - f(x)}{x-a} - f'(a) \\ &= f'(a) - f('a) = 0. \end{aligned}$$

Since the term $\frac{1}{x-a}$ tends to infinity as x tends to a, this suggests that the difference $f(x) - L(x)$ tends to zero so quickly that it overcomes the tendency of $\frac{1}{x-a}$ to tend to infinity. Therefore, the closer x is to a, the better $L(x)$ approximates $f(x)$.

We then noted that we can extend this idea to $f(x,y)$ at (a,b). Assuming $f_x(a,b)$ and $f_y(a,b)$ exist, we set $L(x,y) := f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$, the linear function we expect to be a good approximation of $f(x,y)$ for (x,y) near (a,b). We then said that $f(x,y)$ is *differentiable* at (a,b) if

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{|(x,y) - (a,b)|} = 0,$$

where $|(x,y) - (a,b)| = \sqrt{(x-a)^2 + (y-b)^2}$. We noted then that just as in the case of a single variable, as (x,y) tends to (a,b), the term $\frac{1}{|(x,y)-(a,b)|}$ tends to infinity, so that if the quotient $\frac{f(x,y)-L(x,y)}{|(x,y)-(a,b)|}$ tends to zero, the numerator $f(x,y) - L(x,y)$ tends to zero fast enough to overcome the tendency of $\frac{1}{|(x,y)-(a,b)|}$ to head towards infinity. Thus, we can say that if $f(x,y)$ is differentiable at (a,b), then $L(x,y)$ provides a good approximation of $f(x,y)$ for (x,y) near (a,b). In addition, this approximation improves the closer (x,y) is to (a,b).

We then noted that if $f(x,y)$ is differentiable at (a,b) and Δx and Δy are small, then

$$\begin{aligned} f(a+\Delta x, b+\Delta y) &\cong L(a+\Delta x, b+\Delta y) \\ &= f_x(x,b)(a+\Delta x - a) + f_y(a,b)(b+\Delta y - b) + f(a,b) \\ &= f_x(a,b)\Delta x + f_y(a,b)\Delta y. \end{aligned}$$

We then illustrated this with the following example.

Example. Use the linear approximation to estimate $f(1.01, 2.02)$, for $f(x,y) = x^2 + 2xy$. We assume that $f(x,y)$ is differentiable at $(1,2)$. We calculated $f_x(x,y) = 2x + 2y$, so $f_x(1,2) = 6$; $f_y(x,y) = 2x$, so $f_y(1,2) = 2$. We also have $\Delta z = .01$ and $\Delta y = .02$. Thus,

$$f(1.02, 2.02) \cong f_x(1,2)(.01) + f_y(12)(.02) + f(1,2) = 6(.01) + 2(.02) + 5 = 5.1.$$

Note that $f(1.01, 2.02) = (1.01)^2 + 2(1.01)(2.02) = 5.1005$, so the approximation 5.1 is a good one.

Thursday, February 18. We continued with our discussion of differentiability of functions of two variables. We began by recalling that if $f_x(a,b)$, $f_y(a,b)$ exist and $L(x,y) := f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$, then $f(x,y)$ is differentiable at (a,b) if

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{|(x,y)-(a,b)|} = 0,$$

where $|(x,y)-(a,b)| = \sqrt{(x-a)^2 + (y-b)^2}$. We then easily showed that:

(a) $f(x,y) = x$ and $g(x,y) = y$ are differentiable at any (a,b) and recorded

(b) If $f(x,y)$ and $g(x,y)$ are differentiable at (a,b), then so are:

- (i) $f(x,y) + g(x,y)$, $f(x,y)g(x,y)$, $\frac{f(x,y)}{g(x,y)}$, if $g(a,b) \neq 0$.
- (ii) $h(f(x,y))$, if $h(t)$ is differentiable at $f(a,b)$.

We followed this with a direct limit calculation showing that $f(x,y) = xy$ is differentiable at any point (a,b). A key point was the substitution $x = r\cos(\theta) + a$ and $y = r\sin(\theta) + b$.

We then considered the function $f(x,y) = \begin{cases} \frac{2xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$, and showed:

- (a) $f(x,y)$ is continuous at (0,0): $\lim_{(x,y)\to(0,0)} \frac{2xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r^2\cos(\theta)(\sin(\theta)}{r} = 0 = f(0,0)$.
- (b) $f_x(0,0) = 0 = f_y(0,0)$: $\lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$.
- (c) $L(x,y) = 0$.
- (d) $f(x,y)$ is not differentiable at $(0,0)$:

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{\frac{2xy}{\sqrt{x^2+y^2}} - 0}{\sqrt{x^2+y^2}}$$
$$= \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}.$$

However, this latter limit does not exist, since taking the limit along $x = 0$ gives 0, while taking the limit along $y = x$ gives 1. Thus, the required limit is not 0, so $f(x,y)$ is not differentiable at (0,0).

We then stated the following theorem:

Theorem(Differentiability Criterion). Given $f(x,y)$ and (a,b) in the domain of $f(x,y)$, then $f(x,y)$ is differentiable at (a,b) if the partial derivatives $f_x(x,y)$ and $f_y(x,y)$ exist and are continuous in an open disk containing (a,b).

We finished class by noting that for the function $f(x,y) = \begin{cases} \frac{2xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$, $f_x(x,y)$ is not continuous at (0,0). To see this, we noted that by direct calculation (and item (b) above),

$$f_x(x,y) = \begin{cases} \frac{2y^3}{(x^2+y^2)^{3/2}}, & \text{if } (x,y) = (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}.$$

Noting that $\lim_{(x,y)\to(0,0)} f_x(x,y)$ does not exist shows that $f_x(x,y)$ is not continuous, which in light of the theorem above, explains why $f(x,y)$ is not differentiable at (0,0).

Friday, February 19. We concluded our discussion by recording the fact that if $f(x,y)$ is differentiable at (a,b), then $f(x,y)$ is continuous at (a,b). We gave the following sketch of the reason why: We noted that we must show $\lim_{(x,y)\to(0,0} f(x,y) = f(a,b)$, or that $\lim_{(x,y)\to(0,0)} f(x,y) - f(a,b) = 0$. Then, if we set

$L(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$, we have

$$\lim_{(x,y)\to(a,b)} f(x,y) - f(a,b) = \lim_{(x,y)\to(a,b)} f(x,y) - L(x,y) + \{f_x(a,b)(x-a) + f_y(a,b)(y-b)\}$$
$$= \lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{\sqrt{(x-a)^2 + (y-b)^2}} \cdot \sqrt{(x-a)^2 + (y-b)^2} + \{f_x(a,b)(x-a) + f_y(a,b)(y-b)\}.$$

The last limit is of the form $\lim_{(x,y)\to(a,b)}\{A(x,y)B(x,y) + C(x,y)\}$ where the limit of each of the three terms is zero. Thus, the required limit is zero, so $f(x,y)$ is continuous at (a,b).

We then discussed the chain rule for multivariable functions, starting with the simplest version where $f(x,y) = f(x(t), y(t))$, leading to the formula

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt},$$

and then looking at increasingly more complex cases, leading to the general case, where for the function $f(x_1, x_2, \dots, x_n)$, each $x_i = x_i(u_1, u_2, \dots, u_m)$.This lead to the general version of the chain rule

$$\frac{\partial f}{\partial u_j} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial u_j} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial u_j} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_j}{\partial u_j},$$

for all $j = 1, 2, \dots, m$. We also verified the chain rule in a couple of particular cases, including the cases:

(i) $f(x,y) = x^2y + e^{xy}$, with $x = \sin(t)$ and $y = \cos(t)$.
(ii) $f(x,y) = x^2y$, with $x = uv$ and $y = u^2 + v^2$.

We ended class with an initial discussion of finding extreme values for $f(x,y)$ noting that, in analogy with the case for a function of one variable, a maximum or minimum value of $f(x,y)$ should occur when the tangent plane to the graph of $z = f(x,y)$ is parallel to the xy-plane. Since in this case, such a plane is of the form $z = c$, with c a constant, we noted that this implies $f_x(a,b) = 0 = f_y(a,b)$. We then definied a *critical point* of $f(x,y)$ to be a point (a,b) in the domain of $f(x,y)$ where $f_x(a,b) = 0 = f_y(a,b)$ or a point (a,b) in the domain of $f(x,y)$ where one (or both) of $f_x(a,b)$ or $f_y(a,b)$ are undefined. We then quickly verified that (0,0) is a critical point of $f(x,y) = x^2 + y^2$, as expected from its graph.

Monday, February 22. We began class by reviewing the definition of critical point for a function $f(x,y)$. These were points (a,b) in the domain of $f(x,y)$ for which $f_x(a,b) = 0 = f_y(a,b)$ or for which one of f_x or f_y are undefined at (a,b). We then computed critical points the following examples.

(i) $f(x,y) = x^2 + y^2 - xy - x - 2$, which has critical point $(\frac{2}{3}, \frac{1}{3})$.
(ii) $f(x,y) = -\sqrt{x^2 + y^2} + 2$. Which has (0,0) as a critical point, since the first order partials of $f(x,y)$ are undefined at $(0,0)$. We also noted that the graph of $f(x,y)$ is an inverted cone, with vertex (0,0,2), so that there is a maximum value associated with the critical point (0,0).
(iii) $f(x,y) = x^3 - 3x - y^2 + 4y$, which has critical points (-1,2) and (1,2).

We then defined, what it means for a point to be a relative maximum or relative minimum of $f(x,y)$, or a saddle point on the graph of $f(x,y)$.

If there is an open disk D containing P such that $f(x_0, y_0) \geq f(x,y)$ for all points (x,y) that are in both D and S, then f has a **relative maximum** at P.

If there is an open disk D containing P such that $f(x_0, y_0) \leq f(x,y)$ for all points (x,y) that are in both D and S, then f has a **relative minimum** at P.

Let $P = (x_0, y_0)$ be in the domain of f where $f_x = 0$ and $f_y = 0$ at P. We say P is a **saddle point** of f if, for every open disk D containing P, there are points (x_1, y_1) and (x_2, y_2) in D such that $f(x_0, y_0) > f(x_1, y_1)$ and $f(x_0, y_0) < f(x_2, y_2)$.

This enabled us to then state the Second Derivative Test for critical points of $f(x, y)$:

Theorem 12.8.2 Second Derivative Test

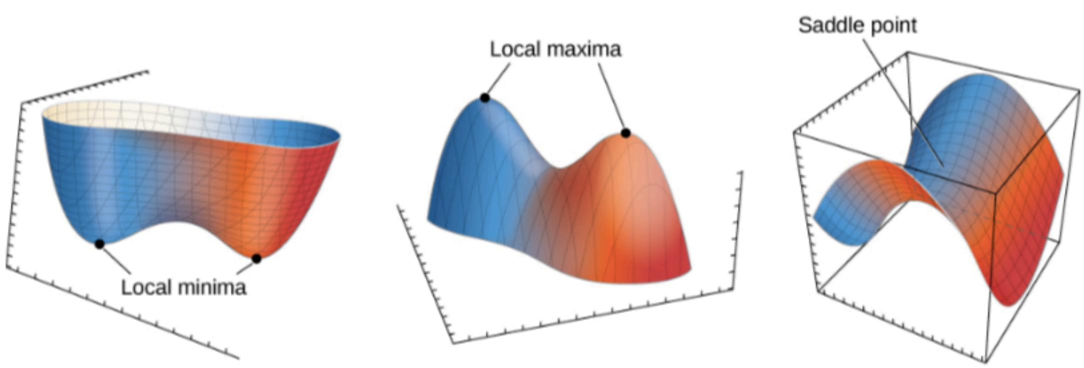
Let R be an open set on which a function $z = f(x, y)$ and all its first and second partial derivatives are defined, let $P = (x_0, y_0)$ be a critical point of f in R, and let

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0).$$

1. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a relative minimum at P.
2. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a relative maximum at P.
3. If $D < 0$, then f has a saddle point at P.
4. If $D = 0$, the test is inconclusive.

Using the second derivative test, we then classified the critical points we found in the examples above:

(i) $f(x, y) = x^2 + y^2 - xy - x - 2$, which has critical point $(\frac{2}{3}, \frac{1}{3})$ that is a relative minimum.
(ii) $f(x, y) = -\sqrt{x^2 + y^2} + 2$. We noted that the second derivative test does not apply.
(iii) $f(x, y) = x^3 - 3x - y^2 + 4y$, which has critical points (-1,2), which gives rise to a relative maximum and (1,2) which gives rise to a saddle point.



We then worked fo the following example: $f(x, y) = x^2 y + y^2 + xy$. Noting first that since $f_x = 2xy + y$ and $f_y = x^2 + 2y + x$, to find the critical points we must solve

$$2xy + y = 0$$
$$x^2 + 2y + x = 0.$$

From the first equation, we have $y = 0$ or $2x + 1 = 0$. If $y = 0$, substituting into the second equation gives $x^2 + x = 0$, so $x = 0$ or $x = -1$. Thus, $(0, 0)$ and (-1,0) are critical points. Assuming $2x + 1 = 0$, we get $x = -\frac{1}{2}$. Substituting this into the second equation yields $(-\frac{1}{2})^2 + 2y + -\frac{1}{2} = 0$, from which we get $y = \frac{1}{8}$. Thus, $(-\frac{1}{2}, \frac{1}{8})$ is also a critical point. We then analyzed these critical points using the second derivative test and the equations: $f_{xx} = 2y$, $f_{yy} = 2$ and $f_{xy} = 2x + 1$.

$D(-1,0) = f_{xx}(-1,0)f_{yy}(-1,0) - f_{xy}(-1,0)^2 = 0 \cdot 2 - (-1)^2 < 0$, so $f(x,y)$ has a saddle point at (-1,0).
$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 0 \cdot 2 - 1^2 < 0$, so $f(x,y)$ has a saddle point at (0,0).
$D(-\frac{1}{2},\frac{1}{8}) = f_{xx}(-\frac{1}{2},\frac{1}{8})f_{yy}(-\frac{1}{2},\frac{1}{8}) - f_{xy}(-\frac{1}{2},\frac{1}{8})^2 = 2(\frac{1}{8}) \cdot 2 - 0^2 > 0$. Since $f_{xx}(-\frac{1}{2},\frac{1}{8}) > 0$, $(-\frac{1}{2},\frac{1}{8})$ gives rise to a relative minimum value of $f(x,y)$.

We ended class by working the following example: Among all rectangular boxes with volume V, find the dimensions of the box with least surface area. For this we let x, y, z denote the dimensions of the box, so that $V = xyz$ and the surface area A is given by $A = 2xy + 2xz + 2yz$. Since $z = \frac{V}{xy}$, we must minimize the function

$$A(x,y) = 2xy + 2x\frac{V}{xy} + 2y\frac{V}{xy} = 2xy + \frac{2V}{y} + \frac{2V}{x}.$$

Note, since no dimension of the box can be zero, the expression for $A(x,y)$ is well defined. We have $A_x = 2y - \frac{2V}{x^2}$ and $A_y = 2x - \frac{2V}{y^2}$, so we must solve:

$$2y - \frac{2V}{x^2} = 0$$
$$2x - \frac{2V}{y^2} = 0.$$

Multiplying the first equation by x and the second equation by y gives

$$2xy - \frac{2V}{x} = 0$$
$$2xy - \frac{2V}{y} = 0.$$

Subtracting these equations yields, $\frac{2V}{y} - \frac{2V}{x} = 0$, from which we see $x = y$. Setting $y = x$ in the first of the displayed equations gives $2x - \frac{2V}{x^2} = 0$. Multiplying by $\frac{x^2}{2}$ gives $x^3 - V = 0$, so $x = \sqrt[3]{V}$. Thus, $y = x = \sqrt[3]{V}$. Finally $z = \frac{V}{xy} = \frac{V}{\sqrt[3]{V} \cdot \sqrt[3]{V}} = \sqrt[3]{V}$. Thus, the rectangular box with fixed volume V and minimal surface area is a cube whose dimensions are $\sqrt[3]{V} \times \sqrt[3]{V} \times \sqrt[3]{V}$.

How do we know the critical point $(\sqrt[3]{V}, \sqrt[3]{V})$ gives rise an absolute minimum value for $A(x,y)$? One can check using the second derivative test, or one can notice that by taking the dimensions of the top of the box to be arbitrarily large, we can take the height of the box sufficiently small to keep the total volume at V. For example, taking $x = 10^4, y = 10^4$ and $z = \frac{V}{10^8}$, we obtain a rectangular box whose total surface area of the box is greater than 10^8, yet the volume remains V. Thus, there can be no upper bound on the surface area of rectangular bixes having volume V.

Tuesday, February 23. After reviewing the process for finding relative extreme values for a function $f(x,y)$ using the second derivative test, as stated in the previous lecture, we noted that one requires that the first and second order partial derivatives of $f(x,y)$ must be continuous in order for the test to be applicable.

We then began a discussion concerning absolute maxima and absolute minima for a function of two variables. We noted that given a subset $X \subseteq \mathbb{R}^2$, $f(x,y)$ has an *absolute maximum* (respectively, *absolute minimum*) at (a,b) if $f(x,y) \leq f(a,b)$ (respectively, $f(a,b) \leq f(x,y)$), for all $(x,y) \in X$. In this case, $f(a,b)$ is the absolute maximum (respectively, absolute minimum) value of $f(x,y)$ on X.

We recalled, that for a function $f(x)$ of one variable, in order to guarantee that $f(x)$ has absolute extreme values, one must assume that $f(x)$ is continuous on a closed interval $[c,d]$. To find these values, one must find critical points on the interior of the interval $[c,d]$ and evaluate $f(x)$ at each of these points, and then one must calculate$f(c)$ and $f(d)$. The largest of these values is the absolute maximum of $f(x)$ on $[c,d]$ and smallest is the absolute minimum of $f(x)$ on $[c,d]$.

We explained that one needs similar condition for functions of two variables. This lead to the following definition.

Definition. Suppose that X is a subset of $\mathbb{R}^2$.

(i) X is *bounded* if there exists a closed disk $D \subseteq \mathbb{R}^2$ with $X \subseteq D$.
(ii) If X is bounded, then X is *closed* if it contained all of its boundary points.

Thus, for example,

(i) The open disk $X_1 = \{0 < x^2 + y^2 < 1\}$ is bounded, but not closed.
(ii) The infinite vertical strip $X_2 = \{(x, y) \mid 0 \leq x \leq 2\}$ is closed, but not bounded.
(iii) The rectangle $X_3 = \{(x, y) \mid -1 \leq x \leq 2,\ -1 \leq y \leq 1\}$ is both closed and bounded.

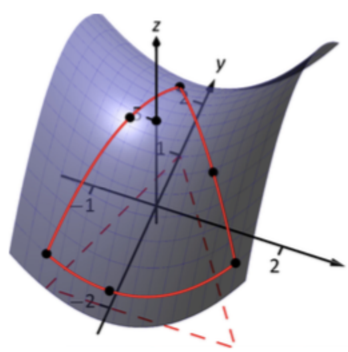
The important theorem concerning absolute extreme values is the following:

Theorem. *Let $X \subseteq \mathbb{R}^2$ be bounded and closed and $f(x, y)$ a continuous function defined on X. Then $f(x, y)$ has both an absolute maximum and absolute minimum value on X.*

We then explained, that the process for finding the absolute extreme values of $f(x, y)$ on X is similar to the one for functions of one variable: First find the critical points on the interior of X, and then find the absolute extreme values of $f(x, y)$ along the boundary of X. The largest and smallest of these values give the required absolute maximum and absolute minimum values of $f(x, y)$ on X. We explained that often, finding the absolute extreme values of $f(x, y)$ along the boundary of X reduces to the one variable case.

After this, we worked through all of the details of the following example, which is Example 12.8.6 in our textbook Calculus 3 by Hartman.

Example. Find the absolute maximum and absolute minimum values for $f(x, y) = x^2 - y^2 + 5$ on the closed triangle X in $\mathbb{R}^2$ with endpoints (0, 1), (-1,-2), (2,-10). In working out the solution, we found as critical points (0,0) in the interior of the triangle, one critical point in the interior of each side of the triangle, and three vertices of the triangle. Substituting these seven critical points into $f(x, y)$ lead to the following: $f(x, y)$ has an absolute maximum value on X of 5.8 at (-1.2, -.8) and an absolute minimum value on X of 1 at (0, -2).



Wednesday, February 24. Today we worked in breakout rooms on the practice problems from the Exam 1 review sheet.

Thursday, February 25. Today we discussed solutions to many of the practice problems on the Exam 1 review sheet. Full solutions were posted on the course webpage.

Thursday, February 26. Exam 1.

Monday, March 1. We began by going over some of the details for problem 1 on Friday's exam. We also noted that a full set of solutions will be posted on our course webpage. We then began a discussion of why the second derivative test works. Recalling Taylor's Theorem from Calculus I, we noted that, given $f(x)$ and $x = a$, under suitable differentiability conditions, a good *quadratic approximation* of $f(x)$ near $x = a$ is given by

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2.$$

We noted that if $f'(a) = 0$, i.e., $f(x)$ has a critical point at $x = a$, then the parabola $y = f(a)+\frac{1}{2}f''(a)(x-a)^2$ approximates $f(x)$ near $x = 0$. This shows that if $f''(a) > 0$, then the graph of $f(x)$ at $x = a$ is concave up, while the graph is concave down if $f''(a) < 0$.

We then stated that a multivariable version of Taylor's Theorem guarantees that, under suitable differentiability conditions, a good quadratic approximation of $f(x, y)$ for (x, y) sufficiently close to (a, b) is given by

$$(*) \quad f(x,y) \approx f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)+\frac{1}{2}\{f_{xx}(a,b)(x-a)^2+2f_{xy}(a,b)(x-a)(y-b)+f_{yy}(a,b)(y-b)^2\}.$$

If we let $Q(x, y)$ denote the right hand side of the expression above, i.e., $Q(x, y)$ is the good quadratic approximation of $f(x, y)$ near (a, b), Taylor's theorem states that

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - Q(x,y)}{|(x,y) - (a,b)|^2} = 0,$$

which is stronger than the condition required for differentiability at (a, b). If we take h_1, h_2 sufficiently small, then (*) yields,

$$(**) \quad f(a+h_1, b+h_2) \approx f(a,b) + f_x(a,b)h_2 + f_y(a,b)h_2 + \frac{1}{2}\tilde{Q}(h_1,h_2),$$

where $\tilde{Q}(h_1, h_2) = f_{xx}(a, b)h_1^2+2f_{xy}(a, b)h_1h_2+f_{yy}(a, b)h_2^2$. We then pointed out that it can be shown that if the conditions $D(a, b) > 0$ and $f_{xx}(a, b) > 0$ hold, then $\tilde{Q}(h_1, h_2) > 0$ for all h_1, h_2 sufficiently small, showing that if $f_x(a, b) = 0 = f_y(a, b)$, then $f(a, b) < f(a + h_1, b + h_2)$ for all h_1, h_2 sufficiently small, implying that $f(x, y)$ has a local minimum at (a, b). Similarly, the other conditions in the second derivative test imply $\tilde{Q}(h_1, h_2) < 0$, for all sufficiently small h_1, h_2 or $\tilde{Q}(h_1, h_2)$ takes both positive and negative values, yielding a relative maximum or saddle point at (a, b). The details of this will be outlined in a bonus problem.

We then discussed a second derivative test for functions $f(x, y, z)$ of three variables. We noted that a point $P = (a, b, c)$ is a critical point if $f(x, y, z)$ if it is either a solution to the system of equations

$$\begin{aligned} f_x(x,y,z) &= 0 \\ f_y(x,y,z) &= 0 \\ f_z(x,y,z) &= 0, \end{aligned}$$

or one of the first order partials is undefined at P. We then defined $D(P) = f_{xx}(P)f_{yy}(P) - f_{xy}(P)^2$ and

$$H(P) = \begin{vmatrix} f_{xx}(P) & f_{xy}(P) & f_{xz}(P) \\ f_{yx}(P) & f_{yy}(P) & f_{yz}(P) \\ f_{zx}(P) & f_{zy}(P) & f_{zz}(P) \end{vmatrix}.$$

$H(P)$ is called the *Hessian* of $f(x, y, z)$ at P.

Second Derivative Test. Suppose $P = (a, b, c)$ critical point that is a solution to the system of equations above and all second order partial derivatives of $f(x, y, z)$ are continuous in a disk about P. Then:

(i) If $f_{xx}(P), D(P), H(P)$ are all greater than zero, $f(x, y, z)$ has a relative minimum at P.
(ii) If $f_{xx}(P) < 0, D(P) > 0, H(P) < 0$, then $f(x, y, z)$ has a relative maximum at P.
(iii) If $H(P) \neq 0$, and neither (i) nor (ii) holds, then $f(x, y, z)$ has a saddlle point at P.
(iv) If $H(P) = 0$, the test is inconclusive.

We then started the example $f(x, y, z) = x^3 + xy + x^2 + y^2 + 3z^2$. We found that $(0, 0, 0)$ and $(-\frac{2}{3}, 0, 0)$ are the only critical points. We will apply the second derivative test in the next lecture.

Tuesday, March 2. We started class with Quiz 3. We then continued with the example from the end of the previous lecture, namely finding and classifying the critical points of the function $f(x, y, z) = x^3 + xy + x^2 + y^2 + 3z^2$. We already know the critical points are (0,0,0) and $(-\frac{2}{3}, 0, 0)$. We have

$$H(x,y,z) = \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 6x+2 & 2y & 0 \\ 2y & 2x+2 & 0 \\ 0 & 0 & 6 \end{vmatrix}$$

At (0,0,0): $H(0,0,0) = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{vmatrix} = 2 \cdot 2 \cdot 6 = 24$. From which we see $f_{xx}(0,0,0) = 2$, since this is the upper left corner of the Hessian and $D(0,0,0) = 4$, since this is the 2×2 determinant of the upper left 2×2 portion of the Hessian. Thus, by the second derivative test, $f(x,y,z)$ has a relative minimum value at (0,0,0).

At $(-\frac{2}{3},0,0)$: $H(-\frac{2}{3},0,0) = \begin{vmatrix} -2 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & 6 \end{vmatrix} = -8 < 0$, $f_{xx}(-\frac{2}{3},0,0) = -2 < 0$ and $D(-\frac{2}{3},0,0) = -\frac{4}{3} < 0$. Since $H(-\frac{2}{3},0,0) \neq 0$, and we are not in case (i) or case (ii) of the second derivative test, $f(x,y,z)$ has a saddle point at $(-\frac{2}{3},0,0)$.

We finished class by noting that one has a general second derivative test for functions $f(x_1,x_2,\ldots,x_n)$ in an arbitrary number of variables. The critical points are found in a similar fashion, e.g., by solving the system of equations

$$\begin{aligned} f_{x_1} &= 0 \\ f_{x_2} &= 0 \\ &\vdots \\ f_{x_n} &= 0. \end{aligned}$$

For a critical point $P = (a_1,a_2,\ldots,a_n)$, one forms the matrix

$$H(P) = \begin{bmatrix} f_{x_1x_1}(P) & f_{x_1x_2}(P) & \cdots & f_{x_1x_n}(P) \\ f_{x_2x_1}(P) & f_{x_2x_2}(P) & \cdots & f_{x_2x_n}(P) \\ \vdots & \vdots & & \vdots \\ f_{x_nx_2}(P) & f_{x_nx_2}(P) & \cdots & f_{x_nx_n}(P) \end{bmatrix}.$$

We then set d_1 to be the upper left 1×1 determinant of $H(P)$, d_2 to be the upper left 2×2 determinant of $H(P)$, ..., d_n to be the full $n \times n$ determinant of $H(P)$. The Second Derivative tests then states, that if the second order partials of $f(x_1,x_2,\ldots,x_n)$ are continuous in an open ball around P, then:

(i) $f(P)$ is a local minimum if $d_1 > 0, d_2 > 0, \ldots, d_n > 0$.
(ii) $f(P)$ is a local maximum if $d_1 < 0, d_2 > 0, d_3 < 0, d_4 > 0, \ldots$.
(iii) f has a saddle point at P if $d_n \neq 0$ and neither case (i) nor case (ii) holds.
(iv) The test is inconclusive if $d_n \neq 0$.

Wednesday, March 3. We began our discussion of constrained maxima and minima problems via the technique of *Lagrange multipliers*. We started by stating the theorem which says the following:

Theorem. Given $f(x,y)$ (or $f(x,yz)$) subject to the constraint $g(x,y) = c$ (or $g(x,y,z) = c$), if :

(i) $P \in \mathbb{R}^2$ (or $P \in \mathbb{R}^3$) is in the domain of f and $g(P) = c$
(ii) f and g have continuous partials at P
(iii) $\nabla g(P) \neq 0$
(iv) $f(P)$ is a maximum value or minimum value of f subject to $g = c$

Then there exists $\lambda \in \mathbb{R}$ such that $\nabla f(P) = \lambda \nabla g(P)$.

The theorem tells us that if the maximum or minimum value we seek exists, then it lies among the points P satisfying $\nabla f(P) = \lambda \nabla g(P)$, for some $\lambda \in \mathbb{R}$. We then worked the following examples.

Example 1. Find the minimal surface area among rectangular boxes with volume 27. The point was that from the practice problems for Exam 1, we know that the required box will be a $3 \times 3 \times 3$ box with minimal surface area 54. The Lagrange method does not require that we solve for one variable (a dimension of the box) in terms of the others. So, we are to minimize $A(x,y,z) = 2xy + 2xz + 2yz$ subject to the constraint that $V(x,y,z) = xyz = 27$. Setting $\nabla A = \lambda \nabla V$ led to the equations:

$$2y + 2z = \lambda yz$$
$$2x + 2z = \lambda xz$$
$$2x + 2y = \lambda xy.$$

Multiplying the first equation by x and the second by y and subtracting leads to $2xz - 2yz = 0$, from which we obtain $y = x$. Using the second and third equation in a similar way leads to $x = z$. So, $x = y = z$. Using this in the constraint equation gives the expected answer $x = y = z = 3$.

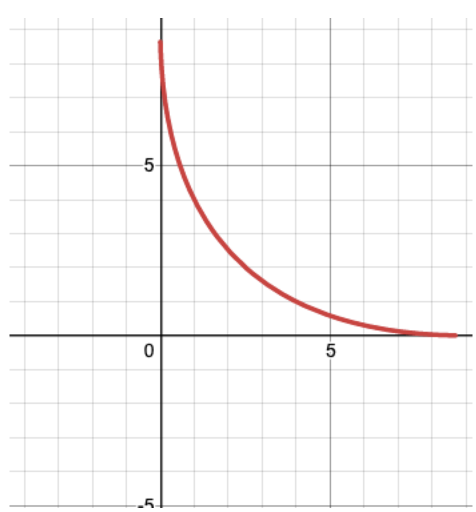
We noted that the next example shows how Lagrange multipliers can be used when one cannot use the constraint equation to solve for one of the variables in terms of the others.

Example 2. Find the extreme values of $f(x,y) = \frac{x^2}{4} + y^2$ subject to the constraint $g(x,y) = x^2 + y^2$. We first noted that since the constraint curve is closed and bounded, the continuous function $f(x,y)$ will have an absolute maximum and absolute minimum value on the curve. We illustrated this by noting the graph of $f(x,y)$ is an elliptic paraboloid, and we are seeking the highest and lowest points on this graph that lie above the unit circle in the xy-plane. Setting $\nabla f = \lambda \nabla g$ leads to

$$\frac{x}{2} = \lambda 2x$$
$$2y = \lambda 2y.$$

We noted that setting $x = 0$ satisfies the first equation, so that $(0,1), (0,-1)$ satisfy the first equation and the constraint equation, no matter what λ is. Since (0,1) and (0,-1) satisfy the second equation with $\lambda = 1$, we have found two critical points. Taking $y = 0$ in the second equation leads to (1,0), and (-1,0) satisfying the constraint equation and the second equation. Taking $\lambda = \frac{1}{4}$ shows that (1,0) and (-1,0) satisfy both equations above, so they are also critical points. We noted that for a given point P we are only looking for at least one value of λ for which P satisfies the required system of equations. Substituting these four critical points into $f(x,y)$ showed that, subject to the constraint, $f(x,y)$ has a maximum value of 1 at $(0,1)$ and (0,-1) and a minimum value of $\frac{1}{4}$ at (1,0) and (-1,0).

Example 3. This example shows that we may have to consider points P for which P is in the domain of f, $g(P) = c$, but ∇f or ∇g may not be defined at P. Take $f(x,y) = 2x + y$, and $g(x,y) = \sqrt{x} + \sqrt{y} = 3$. We noted that the constraint curve below is closed and bounded, since it includes the endpoints (9,0) and (0,9), so that we expect to have a maximum and minimum value for $f(x,y)$ subject to the given constraint.



Setting $\nabla f = \lambda \nabla g$ leads to

$$2 = \frac{\lambda}{2\sqrt{x}}$$
$$1 = \frac{\lambda}{2\sqrt{y}}.$$

Solving each equation for λ leads to $4\sqrt{x} = 2\sqrt{y}$, so $\sqrt{y} = 2\sqrt{x}$. Using this in the constraint equation gives $\sqrt{x} + 2\sqrt{x} = 3$, so $\sqrt{x} = 1$ and thus $x = 1$. Since $\sqrt{y} = 2\sqrt{x}$ and $x = 1$, we get $\sqrt{y} = 2$, so $y = 4$. Thus, (1,4) is a critical point, and $f(1,4) = 6$. However, (9,0) and (0,9) are in the domain of $f(x,y)$ and meet the constraint requirement, and so we must test these values as well, since ∇g is undefined at these points. We have $f(9,0) = 18$ and $f(0,9) = 9$. Thus, the minimum value of $f(x,y)$ subject to the constraint is 6 and the maximum value is 18.

Thursday, March 4. We continued our discussion of Lagrange multipliers, starting with the following example.

Example 1. Find the extreme values of $f(x,y,z) = x - y + z$ subject to $g(x,y,z) = x^2 + y^2 + z^2 = R^2$. Taking gradients and setting $\nabla f = \lambda \nabla g$ led to the equations

$$1 = \lambda 2x$$
$$-1 = \lambda 2y$$
$$1 = \lambda 2z.$$

These equations imply $\lambda = \frac{1}{2x} = -\frac{1}{2y} = \frac{1}{2z}$, so $y = -x$ and $z = x$. From the constraint equation we have $x^2 + (-x)^2 + x^2 = R^2$, so $3x^2 = R^2$, and therefore $x = \pm\frac{R}{\sqrt{3}}$. The critical points are $(\frac{R}{\sqrt{3}}, -\frac{R}{\sqrt{3}}, \frac{R}{\sqrt{3}})$, which yields a maximum value of $\frac{3R}{\sqrt{3}}$, and $(-\frac{R}{\sqrt{3}}, \frac{R}{\sqrt{3}}, -\frac{R}{\sqrt{3}})$, which yields a minimum value of $-\frac{3R}{\sqrt{3}}$.

We then gave an indication of why the method of Lagrange multipliers works, in the special case where we seek the extreme values of $f(x,y)$ subject to a constraint whose curve is given by a function of x, i.e., we take $g(x,y) = y - h(x) = 0$ as the constraint equation. In this case $\nabla g = -h'(x)\vec{i} + \vec{j}$. The extreme values of $f(x,y)$ along the curve correspond to the critical points of $F(x) = f(x, h(x))$. If we set $F'(x) = 0$, we get,

$$\frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dh(x)}{dx} = 0.$$

Thus, if $F'(x_0) = 0$, and $y_0 = h(x_0)$, then $\frac{\partial f}{\partial x}(x_0,y_0) = -h'(x_0) \cdot \frac{\partial f}{\partial y}(x_0,y_0)$. Thus,

$$\begin{aligned}
\nabla f(x_0,y_0) &= \frac{\partial f}{\partial x}(x_0,y_0)\vec{i} + \frac{\partial f}{\partial y}(x_0,y_0)\vec{j} \\
&= -h'(x_0) \cdot \frac{\partial f}{\partial y}(x_0,y_0)\vec{i} + \frac{\partial f}{\partial y}(x_0,y_0)\vec{j} \\
&= \frac{\partial f}{\partial y}(x_0,y_0)\{-h'(x_0)\vec{i} + \vec{j}\} \\
&= \frac{\partial f}{\partial y}(x_0,y_0) \cdot \nabla g(x_0,y_0),
\end{aligned}$$

which shows that $\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0)$, for $\lambda = \frac{\partial f}{\partial y}(x_0,y_0)$,which is what we expect.

We then worked the following example which shows that the technique of Lagrange multipliers can be used to find the absolute maximum and minimum values of a continuous function on a closed and bounded region in Euclidean space. The point being, for example, if the region lies in $\mathbb{R}^2$, we no longer need a region whose boundary is given (possibly piecewise) by a function expressing one of the variables x or y in terms of the other.

Example 2. Find the absolute maximum and absolute minimum values of $f(x,y) = xy$ over the region $D : 0 \leq x^2 + y^2 \leq 1$. As we did in the lecture on February 23, we first find critical points in the interior of

D, in the usual way, by solving

$$f_x = y = 0$$
$$f_y = x = 0,$$

which shows that (0,0) is the only critical point in the interior of D. For points on the boundary of D, taking $g(x,y) = x^2 + y^2 = 1$ as the constraint, if we set $\nabla f = \lambda \nabla g$, we get

$$y = \lambda 2x$$
$$x = \lambda 2y,$$

from which it follows that $\frac{y}{2x} = \lambda = \frac{x}{2y}$, and thus, $x^2 = y^2$, so $y = \pm x$. From the constraint equation, we have $x^2 + x^2 = 1$, so that $x = \pm\frac{\sqrt{2}}{2}$. Thus, we have four critical points: $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}), (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}), (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, the first two of which give a maximum value of $f(x,y)$ on the boundary of D of $\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{1}{2}$, while the second two critical points give a minimum value on the boundary of $-\frac{1}{2}$. Since $f(0,0) = 0$, and $-\frac{1}{2} < 0 < \frac{1}{2}$, the absolute maximum of $f(x,y)$ on D is $\frac{1}{2}$ and the absolute minimum value is $-\frac{1}{2}$.

We ended class by considering the situation with more than one constraint. We mentioned that in this case, if one wants to find the extreme values of f subject to constraints, $g_1 = c_1, \dots, g_n = c_n$, then one has to consider the system of equations arising from the vector equation $\nabla f = \lambda_1 \nabla g_1 + \cdots + \lambda_n \nabla g_n$, together with the constraint equations. We then worked the following example.

Example 3. Find the extreme values of $f(x,y,z) = x + y + z$ subject to $g_1(x,y,z) = x^2 + y^2 = 2$ and $g_2(x,y,z) = x + z = 1$. We began by noting that the first constraint equation shows that x and y are bounded, and thus, the second equation shows that z is bounded, so that the constraints together define a closed and bounded subset of $\mathbb{R}^3$. Thus, we can expect a maximum and minimum value of $f(x,y,z)$ subject to the given constraints. Setting $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$ led to the equations

$$1 = \lambda_1 \cdot 2x + \lambda_2$$
$$1 = \lambda_1 \cdot 2y$$
$$1 = \lambda_2.$$

Subtracting the third equation from the first gives $0 = 2\lambda_1 x$, so $x = 0$, since the second equation implies $\lambda_1 \neq 0$. From the first constraint equation we get $0^2 + y^2 = 2$, so $y = \pm\sqrt{2}$. From the second constraint equation we get $0 + z = 1$, so $z = 1$. Therefore, we have critical points $P_1 = (0, \sqrt{2}, 1)$ and $P_2 = (0, -\sqrt{2}, 1)$. Substituting into $f(x,y,z)$ we have $f(P_1) = \sqrt{2} + 1$, and $f(P_2) = -\sqrt{2} + 1$. Thus $\sqrt{2} + 1$ is the maximum value of $f(x,y,z)$ subject to the given constraints, and $-\sqrt{2} + 1$ is the corresponding minimum value.

Friday, March 5. We did group work in Breakout Rooms on the following problems:

1. Find the extreme values of $f(x,y) = x - y$, subject to $x^2 - y^2 = 2$.

2. Find the absolute maximum and absolute minimum values of $f(x,y) = x^2 + xy + y^2$ on the closed and bounded region $0 \leq x^2 + y^2 \leq 1$.

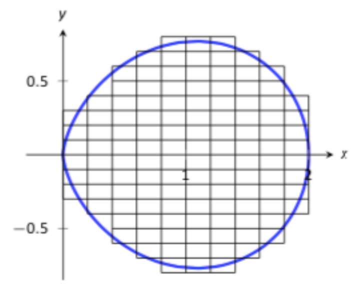
3. Find the absolute maximum and absolute minimum values of $f(x,y,z) = xyz$ on the solid sphere $S : 0 \leq x^2 + y^2 + z^2 \leq 81$.

4. Find the extreme values $f(x,y,x) = x + y + z$ subject to $x^2 + y^2 = 1$ and $2x + z = 1$.

5. Find three positive numbers whose product is 27 and whose sum is minimal. Then find three positive numbers whose sum is 27 and whose product is maximal.

Monday, March 8. We began our discussion of double integration by recalling the definition of $\int_a^b f(x)\,dx$ as a limit of Riemann sums $\sum_i f(c_i)\Delta x_i$ over increasingly finer partitions of the interval $[a,b]$. We observed that the notation $\int_a^b f(x)\,dx$ is suggestive if the process: $\int$ is an elongated "S" for sum, and $f(x)\,dx$, values of $f(x)$ times small increments in x, represents what is being summed. We also noted that $\int_a^b f(x)\,dx$ depends on the domain of integration, $[a,b]$ and the integrand $f(x)$, and that the same will hold for double integrals.

We then began a discussion of what $\int\int_D f(x,y)\ dA$ should mean, where $R \subseteq \mathbb{R}^2$ is a possible domain of integration. Proceeding by analogy, we observed that the notation is suggestive: we should be summing (via a double sum) values of $f(x,y)$ times small increments of area. For this we described the process of covering the region R with small rectangles $\Delta x_i \times \Delta y_j$, something like this:



We selected a point (c_i, d_j) from each $\Delta x_1 \times \Delta y_j$ rectangle and formed the Riemann sum $\sum_i \sum_j f(c_i, d_j)\Delta x_i \Delta y_j$. We then defined

$$\int\int_D f(x,y)\ dA = \lim_{\Delta x, \Delta y \to 0} \sum_i \sum_j f(c_i, d_j)\Delta x_i \Delta y_j,$$

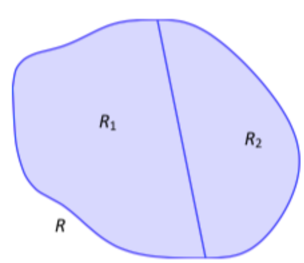
provided the limit exists. The resulting number is called the *double integral of f(x,y) over the region R*. We then stated the following important theorem:

If $f(x,y)$ is continuous on R, and R is a closed and bounded subset of $\mathbb{R}^2$, then $\displaystyle\int\int_D f(x,y)\ dA$ exists.

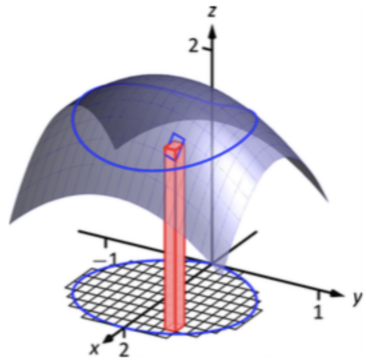
We followed this by recording by the expected properties of the double integral:

Properties. Assume $f(x,y)$, $g(x,y)$ are integrable over the region R. Then:

(i) $\int\int_R \{f(x,y) \pm g(x,y)\}\ dA = \int\int_R f(x,y)\ dA \pm \int\int_R g(x,y)\ dA$.
(ii) $\int\int_R \lambda \cdot f(x,y)\ dA = \lambda \cdot \int\int_D f(x,y)\ dA$, for all $\lambda \in \mathbb{R}$.
(iii) $\int\int_D f(x,y)\ dA = \int\int_{R_1} f(x,y)\ dA + \int\int_{R_2} f(x,y)\ dA$, where $R = R_1 \cup R_2$ and either R_1, R_2 are disjoint, or only intersect along their boundaries



We ended class by noting that $\int\int_R\ dA = \text{area}(R)$ and, if $f(x,y) \geq 0$, for all $(x,y) \in R$, then $\int\int_D f(x,y)\ dA$ represents the volume of the region in $\mathbb{R}^3$ bounded above by the graph of $f(x,y)$ and bounded below by R.



(b)

This follows since in the Riemann sum, the terms $f(c_i, d_j)\Delta x_j \Delta y_j$ are the volumes of the rectangular boxes whose bases are the rectangles $\Delta x_i \times \Delta y_j$ with heights $f(x_i, y_j)$. Thus, the full sum $\sum_i \sum_j f(c_i, d_j)\Delta x_i \Delta y_j$ approximates the required volume, and these approximations improve as $\Delta x, \Delta y \to 0$. Thus, in the limit, we get the volume under the graph of $f(x, y)$ and above R.

Tuesday, March 9. We began class with Quiz 4. We then reviewed the definition of $\int\int_D f(x, y)\ dA$ and calculated a few Riemann sums for $\int\int_R xy\ dA$, where $R = [0, 1] \times [0, 1]$, the unit square in the xy-plane. Taking a Riemann sum with just one term, using the center of R, we obtained $\frac{1}{4}$. Further subdivisions with halves and quarters, and choosing center points of each su-brectangle, also lead to $\frac{1}{4}$. So we expected $\int\int_R xy\ dA = \frac{1}{4}$. We then stated the following theorem:

Fubini's Theorem for rectangles. Suppose $f(x, y)$ is continuous on the rectangle $R = [a, b] \times [c, d]$. Then

$$\int\int_D f(x, y)\ dA = \int_c^d \{\int_a^b f(x, y)\ dx\}\ dy = \int_a^b \{\int_c^d f(x, y)\ dy\}\ dx,$$

where $\int_c^d \{\int_a^b f(x, y)\ dx\}\ dy$ and $\int_a^b \{\int_c^d f(x, y)\ dy\}\ dx$ are *iterated integrals*. We discussed how to calculate iterated integrals and verified Fubini's theorem for $\int\int_R xy\ dA$, with R the unit square in the xy-plane, obtaining the expected value of $\frac{1}{4}$.

Wednesday, March 10. We continued our discussion of calculating doubles integrals as iterated integrals. We began by calculating $\int\int_R x^2y + 2xy + 7\ dA$ in two ways, for $R = [-1, 1] \times [1, 2]$. We saw

$$\begin{aligned}\int\int +x^2y + 2xy + 7\ dA &= \int_{-1}^1 \{\int_1^2 x^2y + 2xy + 7\ dy\}\ dx = 15\\ &= \int_1^2 \{\int_{-1}^1 x^2y + 2xy + 7\ dx\}\ dy = 15.\end{aligned}$$

We then asked the class to verify on their own that

$$\int\int_R x^2y + 2xy + 7\ dA = \int\int_R x^2y\ dA + 2\int\int_R xy\ dA + 7\int\int_R\ dA.$$

Next we stated the version of Fubini's theorem that enables us to calculated double integrals over more general regions:

Theorem 13.2.2 Fubini's Theorem

Let R be a closed, bounded region in the x-y plane and let $z = f(x, y)$ be a continuous function on R.

1. If R is bounded by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where g_1 and g_2 are continuous functions on $[a, b]$, then

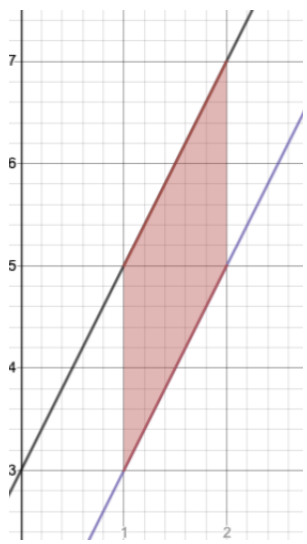
$$\iint_R f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx.$$

2. If R is bounded by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, where h_1 and h_2 are continuous functions on $[c, d]$, then

$$\iint_R f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy.$$

We followed this by working (or setting up) several examples, noting that the important part of each example is setting up the iterated integral correctly.

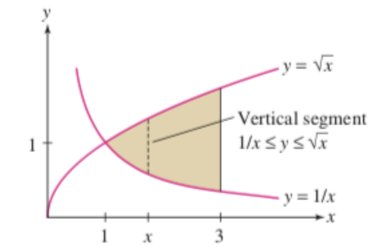
Example 1. Calculate $\int\int 3x^2 + 2y\ dA$, for the region R of Type 1



with $1 \leq x \leq 2$ and $2x + 1 \leq y \leq 2x + 3$. This lead to

$$\begin{aligned}
\int\int_R 3x^2 + 2y\ dA &= \int_1^2 \int_{2x+1}^{2x+3} 3x^2 + 2y\ dy\ dy \\
&= \int_1^2 \{3x^2 y + y^2\}_{y=2x+1}^{y=2x+3}\ dx \\
&= \int_1^2 (3x^2(2x+2) + (2x+2)^2) - (3x^2(2x+1) + (2x+1)^2)\ dx \\
&= \int_x^2 6x^2 + 8x + 8\ dx \\
&= (2x^3 + 4x^2 + 8x)_1^2 \\
&= (16 + 16 + 16) - (2 + 4 + 8) = 18.
\end{aligned}$$

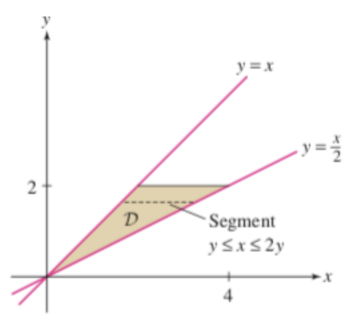
Example 2. Calculate $\int\int_R x^2y\ dA$, for the region R of Type 1



with $1 \le x \le 3$ and $\frac{1}{x} \le y \le \sqrt{x}$.

$$\begin{aligned}
\int\int_R x^2y\ dA &= \int_1^3 \int_{\frac{1}{x}}^{\sqrt{x}} x^2y\ dy\ dx \\
&= \int_1^2 (\frac{1}{2}x^2y^2)_{y=\frac{1}{x}}^{y=\sqrt{x}}\ dx \\
&= \int_1^3 \frac{1}{2}\{x^2(\sqrt{x})^2 - x^2(\frac{1}{x})^2\}\ dx \\
&= \frac{1}{2}\int_1^3 x^3 - 1\ dx \\
&= \frac{1}{2}(\frac{1}{4}x^4 - x)_1^3 \\
&= \frac{1}{2}\{(\frac{81}{4} - 3) - (\frac{1}{4} - 1)\} = 9.
\end{aligned}$$

Example 3. Set up $\int\int_R 2x + 4\ dA$ in two different ways for the region R



.

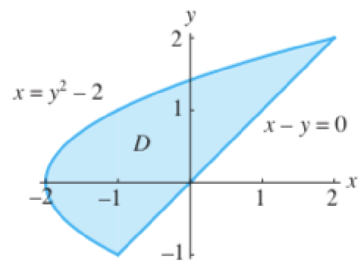
We first noted, that we can view R as a region of Type 2 with $0 \le y \le 2$ and $y \le x \le 2y$. Thus,

$$\int\int_R 2x + 4\ dA = \int_1^2 \int_y^{2y} 2x + 4\ dx\ dy.$$

On the other hand, R is not bounded above by a single function of x. In terms of regions of Type 1, $R = R_1 \cup R_2$, where R_1 is given by $0 \le x \le 2$ and $\frac{x}{2} \le y \le x$ and R_2 is given by $2 \le x \le 4$ and $\frac{x}{2} \le y \le 2$. Thus,

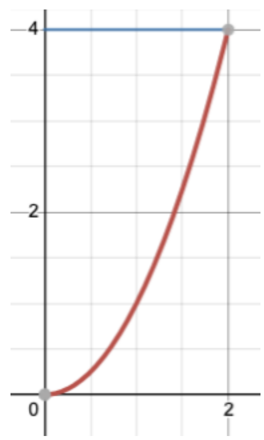
$$\int\int_R 2x + 4\ dA = \int_0^2 \int_{\frac{x}{2}}^{x} 2x + 4\ dy\ dx + \int_2^4 \int_{\frac{x}{2}}^{2} 2x + 4\ dy\ dx.$$

Thursday, March 11. We continued working examples of double integrals, the first being carrying out both calculations from Example 3 in the previous lecture, the answer being 16. We then had the class take a few minutes to set up the following integral : $\int\int_R y\ dA$ for R



We noted that R is a region of Type 2, and thus one needs to find the y coordinates of the points where the two graphs intersect. This required solving $y = y^2 - 2$, so that $y = -1$ and $y = 2$. The required integral is $\int_{-1}^{2}\int_{y^2-2}^{y} y\ dx\ dy$. We finished class with the following example.

Example. Calculate $\int\int_R x\cos(y^2)\ dA$, for R given by $0 \le x \le 2$ and $x^2 \le y \le 4$,



We noted that R is a region of Type 1 and also a region of Type 2. Since one cannot integrate $x\cos(y^2)$ with respect to y, we cannot solve $\int\int_R x\cos(y^2)\ dA\ dA = \int_0^2\int_{x^2}^4 x\cos(y^2)\ dy\ dx$. As a region of Type 2, R can be described by $0 \le y \le 4$ and $0 \le x \le \sqrt{y}$. Thus:

$$
\begin{aligned}
\int\int_R x\cos(y^2)\ dA &= \int_0^2\int_0^{\sqrt{y}} x\cos(y^2)\ dx\ dy \\
&= \int_0^4 \{\frac{x^2}{2}\cos(y^2)\}_0^{x=\sqrt{y}}\ dy \\
&= \int_0^4 \frac{(\sqrt{y})^2}{2}\cos(y^2) - 0\ dy \\
&= \frac{1}{2}\int_0^4 y\cos(y^2)\ dy \\
&= \frac{1}{4}\{\sin(y^2)\}_0^4 \\
&= \frac{1}{4}\sin(16).
\end{aligned}
$$

Friday, March 12. We continued with examples of double integration, starting with the bonus problem assigned at the end of Thursday's class.

Example 1. Set up $\int\int_R e^{-x^2}\ dA$ in two ways, for R the region given by $0 \le x \le 2$ and $0 \le y \le 2x$. We have

$$\begin{aligned}\int\int_R e^{-x^2}\ dA &= \int_0^2 \int_0^{2x} e^{-x^2}\ dy\ dx\\ &= \int_0^4 \int_{\frac{y}{2}} e^{-x^2}\ dx\ dy.\end{aligned}$$

We noted it is not possible to calculate the second integral. Calculating the first integral leads to

$$\begin{aligned}\int_0^2 \int_0^{2x} e^{-x^2}\ dy\ dx &= \int_0^2 \{e^{-x^2} y\}_0^{y=2x}\ dx\\ &= \int_0^2 2xe^{-x^2}\ dx\\ &= \{-e^{-x^2}\}_0^2\\ &= -e^{-4} + 1.\end{aligned}$$

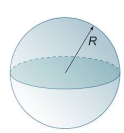
We then discussed average value for functions of two variables, by first recalling the formula from Calculus 2 giving the average value of a function of one variable over a closed interval. We explained in general terms how if $f(x)$ is positive on the interval $[a, b]$, then the area of the region bounded above by the graph of $f(x)$ and below by $[a, b]$ will be the same as the area of a box with base $[a, b]$ and height $f(c)$ for some c. Since this means $\int_a^b f(x)\ dx = f(c) \cdot (b - a)$, it follows that $f(c) = \frac{1}{b-a}\int_a^b f(x)\ dx$, which we recognize as the average value of $f(x)$ on $[a, b]$. We then defined the *average value* of $f(x, y)$ over the region bounded R as

$$\text{Average value} = \frac{1}{\text{area}(R)} \int\int_D f(x, y)\ dA.$$

We calculated the average value of $f(x, y) = x^2 + 2xy$ over the rectangle $R = [-1, 1] \times [-2, 2]$. Since the area of R is 8, the average value of $f(x, y)$ over R is given by $\frac{1}{8}\int_{-2}^2 \int_{-1}^2 x^2 + 2xy\ dx\ dy$, which was easily seen to be $\frac{1}{3}$.

We then recalled that if $f(x, y)$ is positive over a region R, then $\int\int_D f(x, y)\ dA$ represents the volume of the region in $\mathbb{R}^2$ bounded above by the graph of $f(x, y)$ and bounded below by the region R. We then verified that the volume of a rectangular box having dimensions $a \times b \times c$ is abc by calculating the double integral $\int_0^b \int_0^a c\ dx\ dy$.

We then turned to the question of calculating the volume of the sphere of radius R centered at the origin. (In class, we used ρ for the radius.)



We can integrate the function $f(x, y) = \sqrt{R^2 - x^2 - y^2}$ over the closed disk $D : 0 \le x^2 + y^2 \le R^2$. This will give us the volume of the top half of our sphere. We can think of D as a region of Type 2, being bounded above by the curve $y = \sqrt{R^2 - x^2}$ and bounded below by the curve $y = -\sqrt{R^2 - x^2}$, with $-R \le x \le R$.

Thus, the volume of the sphere is given by

$$2\int\int_D \sqrt{R^2 - x^2 - y^2}\ dA = 2\int_{-R}^R \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \sqrt{R^2 - x^2 - y^2}\ dy\ dx.$$

Note that the first of the two iterated integrals requires consideration of an indefinite integral of the form $\int \sqrt{a^2 - y^2}\ dy$, where $a = \sqrt{R^2 - x^2}$.

This can be worked out using a trig substitution like $y = a\sin(u)$, and the answer becomes

$$\frac{y}{2}\sqrt{a^2 - y^2} + \frac{a^2}{2}\sin^{-1}(\frac{y}{2}).$$

We must then replace a by $\sqrt{R^2 - x^2}$, take the difference of y evaluated at $\sqrt{R^2 - x^2}$ and $-\sqrt{R^2 - x^2}$, and then integrate with respect to x.

There is a better solution!

The idea is a two variable form of u-substitution, namely, we use polar coordinates as follows: Set $x = r\cos(\theta)$, $y = r\sin(\theta)$, $dA = rdrd\theta$. We will explain this latter equality, in a future lecture, but the point is that just like in u-substitution, we don't simply exchange dx for du, here we do not simply exchange dA for $drd\theta$, as there is a scaling factor of r involved. In terms of r and θ, D is described as: $0 \le r \le R$ and $0 \le \theta \le 2\pi$. Upon substituting, we get:

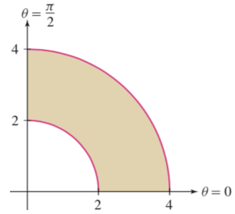
$$\begin{aligned}
2\int\int_D \sqrt{R^2 - x^2 - y^2}\ dA &= 2\int_0^{2\pi}\int_0^R \sqrt{R^2 - (r\cos(\theta))^2 - (r\sin(\theta))^2}\ r\ dr\ d\theta \\
&= 2\int_0^{2\pi}\int_0^R \sqrt{R^2 - r^2(\cos^2(\theta) + \sin^2(\theta))}\ r\ dr\ d\theta \\
&= 2\int_0^{2\pi}\int_0^R r\sqrt{R^2 - r^2}\ dr\ d\theta
\end{aligned}$$

Note that now the domain of integration is a rectangle in the (r, θ) plane. An easy u-substitution shows that $\int r\sqrt{R^2 - r^2}\ dr = -\frac{1}{3}(R^2 - r^2)^{\frac{3}{2}}$. Thus:

$$\begin{aligned}
2\int_0^{2\pi}\int_0^R r\sqrt{R^2 - r^2}\ dr\ d\theta &= 2\int_0^{2\pi} -\frac{1}{3}(R^2 - r^2)^{\frac{3}{2}}\Big|_{r=0}^{r=R}\ d\theta \\
&= 2\int_0^{2\pi} 0 + \frac{R^3}{3}\ d\theta \\
&= 2\cdot\frac{R^3}{3}\theta\Big|_{\theta=0}^{\theta=2\pi} \\
&= 2\cdot\frac{2\pi R^3}{3} \\
&= \frac{4\pi}{3}R^3.
\end{aligned}$$

We then considered the following example.

Example 2. Calculate $\int\int_R x + y\ dA$, where D is the region



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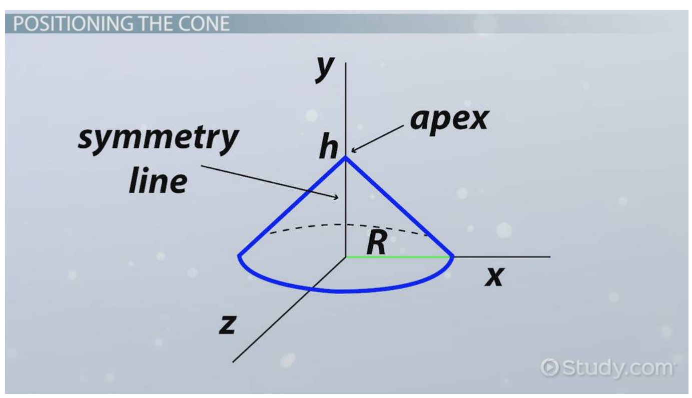
We noted that, in polar coordinates, D can be described as $0 \le \theta \le \frac{\pi}{2}$ and $2 \le r \le 4$. Thus:

$$\begin{aligned}\int\int_D x+y\ dA &= \int_0^{\frac{\pi}{2}}\int_2^4 (r\cos(\theta)+r\sin(\theta))r\ dr\ d\theta \\ &= \int_0^{\frac{\pi}{2}}\int_2^4 r^2(cos(\theta)+\sin(\theta))\ dr\ d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{r^3}{3}(\cos(\theta)+\sin(\theta))\Big|_{r=2}^{r=4} d\theta \\ &= \int_0^{\frac{\pi}{2}} (\frac{64}{3}-\frac{8}{3})\cdot(\cos(\theta)+\sin(\theta))d\theta \\ &= \frac{56}{3}\cdot(\sin(\theta)-\cos(\theta))\Big|_0^{\frac{\pi}{2}} \\ &= \frac{56}{3}\cdot\{(1-0)-(0-1)\} \\ &= \frac{112}{3}.\end{aligned}$$

Monday, March 15. We continued our discussion of calculating double integrals via polar coordinates, with an emphasis on the fact that $\int\int_D f(x,y)\ dA$ is a quantity depending upon the domain $D \subseteq \mathbb{R}^2$ and the function $f(x,y)$ and that iterated integrals, whether using rectangular or polar coordinates, are just means to calculate this quantity. We began with the following example.

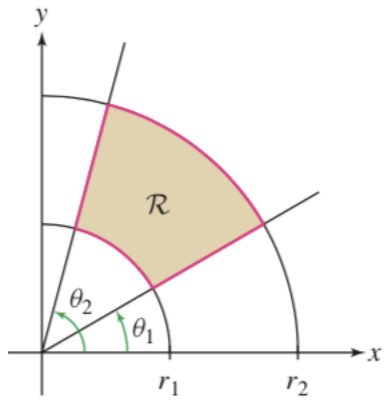
Examplel 1. Calculate the volume of that portion of the inverted cone $z = h - \sqrt{x^2+y^2}$, with $z \ge 0$.



For our example, $R = h$, since $z = 0 = h - \sqrt{x^2+y^2}$ implies $h^2 = x^2+y^2 = R^2$. So the domain of integration D is the disk of radius h in the xy-plane. Thus, we have

$$\begin{aligned}\text{Volume} &= \int\int_D h\sqrt{x^2+y^2}\ dA \\ &= \int_0^{frm-epi}\int_0^h \{h - \sqrt{(r\cos(\theta))^2+(r\sin(\theta)^2}\}\ r\ dr\ d\theta \\ &= \int_0^{2\pi}\int_0^h hr - r^2\ dr\ d\theta\ = \int_0^{2\pi}(h\frac{r^2}{2}-\frac{r^3}{3})\Big|_{r=0}^{r=h} d\theta \\ &= \int_0^{2\pi}\frac{h^3}{2}-\frac{h^3}{3}\ d\theta = \int_0^{2\pi}\frac{h^3}{6}\ d\theta \\ &= \frac{h^3}{6}\theta\Big|_0^{2\pi} = \frac{h^3}{6}(2\pi-0) = \frac{\pi}{3}h^3.\end{aligned}$$

We then began a discussion of why we use $dA = r\ dr\ d\theta$ in the polar coordinate version of $\int\int_D f(x,y)\ dA$. The point is that we may subdivide D into regions of our choosing. When we subdivide the domain of integration D into small polar rectangles with $\theta_1 \leq \theta \leq \theta_2$ and $r_2 \leq r \leq r_2$, D is covered by regions $\mathcal{R}$ that look like:



When we form a Riemann sum, we must multiply a function value on $\mathcal{R}$ by the area of $\mathcal{R}$. The area of $\mathcal{R}$ is:

$$\frac{r_2^2}{2} \cdot (\theta_2 - \theta_1) - \frac{r_1^2}{2} \cdot (\theta_2 - \theta_1).$$

Now set $\theta_2 - \theta_1 = \Delta\theta$, $r = r_1$ and $r_2 = r + \Delta r$. Then:

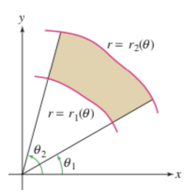
$$\begin{aligned}\text{area}(\mathcal{R}) &= \frac{(r+\Delta r)^2}{2} \cdot \Delta\theta - \frac{r^2}{2} \cdot \Delta\theta \\ &= r\ \Delta r\ \Delta\theta + \frac{(\Delta r)^2 \Delta\theta}{2}.\end{aligned}$$

When Δr and $\Delta\theta$ are small, the term $\frac{(\Delta r)^2 \Delta\theta}{2}$ is much smaller than the term $r\ \Delta r\ \Delta\theta$. Thus:

$$\text{area}(\mathcal{R}) \approx r\ \Delta r\ \Delta\theta.$$

This approximation gets better as Δr and $\Delta\theta$ tend to zero. Thus dA, measured in polar coordinates, becomes $r\ dr\ d\theta$. We can use these approximations in the Riemann sums defining the double integral, which in the limit becomes an iterated integral $\int\int f(r\cos(\theta), r\sin(\theta))\ r\ dr\ d\theta$.

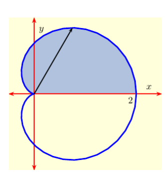
We then showed how to integral over more general polar regions. Suppose we wish to integrate the continuous function $f(x,y)$ over a region D of the following type:



Here D is given by $r_1(\theta) \leq r \leq r_2(\theta)$ and $\theta_1 \leq \theta \leq \theta_2$, where each $r_i(\theta)$ is a function of θ. Then we have

$$\int\int_D f(x,y)\ dA = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos(\theta), r\sin(\theta))\ r dr d\theta.$$

Example 2. Calculate $\int\int_D y\ dA$, where D is the set of points lying above the x-axis and inside the cardioid $r = 1 + \cos(\theta)$:



Solution:

$$\int\int_D y\ dA = \int_0^{\pi}\int_0^{1+\cos(\theta)} r\sin(\theta) r dr d\theta$$
$$= \int_0^{\pi}\int_0^{1+\cos(\theta)} r^2\sin(\theta)\ dr d\theta$$
$$= \int_0^{\pi} \frac{r^3}{3}\Big|_0^{1+\cos(\theta)} \sin(\theta)\ d\theta$$
$$= \frac{1}{3}\int_0^{\pi}(1+\cos(\theta))^3\sin(\theta)\ d\theta$$

The u-substitution $u = 1+\cos(\theta)$ yields

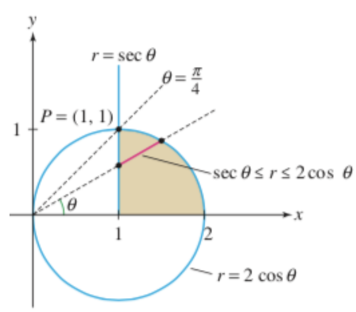
$$\int (1+\cos(\theta))^3\sin(\theta)\ d\theta = \int -u^3\ du = -\frac{1}{4}u^4 = -\frac{1}{4}(1+\cos(\theta))^4.$$

Thus:

$$\int\int_D y\ dA = \frac{1}{3}\{-\frac{1}{4}(1+\cos(\theta))^4\Big|_{\theta=0}^{\theta=\pi}\}$$
$$= -\frac{1}{12}\cdot\{(1+-1)^4-(1+1)^4\}$$
$$= \frac{16}{12}$$
$$= \frac{4}{3}.$$

We ended class by setting up the double integral in polar coordinates for the following example.

Example 3. Find $\int\int_D (x^2+y^2)^{-2}\ dA$, where D is the shaded region:



.

Solution: From the diagram, we see

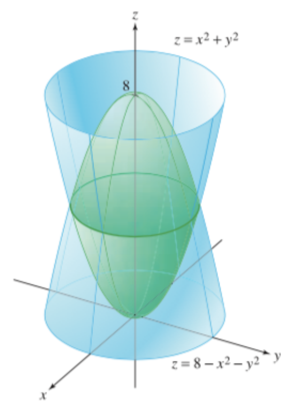
$$\int\int_D (x^2+y^2)^{-2}\ dA = \int_0^{\frac{\pi}{4}}\int_{\sec(\theta)}^{2\cos(\theta)} (r^2\cos^2(\theta)+r^2\sin^2(\theta))^{-2}\ r dr d\theta$$

Tuesday, March 16. We started class with Quiz 5. Afterwards, we worked examples of the following type: If $f(x,y)$ and $g(x,y)$ are continuous over a domain $D\subseteq\mathbb{R}^2$ and $f(x,y)\geq g(x,y)$, for all $(x,y)\in D$, then the volume of the region in $\mathbb{R}^3$ bounded above by the graph of $f(x,y)$ and bounded below by the graph of $g(x,y)$ over the region D is $\int\int_D f(x,y)-g(x,y)\ dA$.

Example 1. Find the volume bounded by the paraboloids $z = 8-x^2-y^2$ and $z = x^2+y^2$, over the region $D : [-1,1]\times[-1.1]$.

Solution: Since $z = 8-x^2-y^2$ has vertex at $(0,0,8)$ and $z = x^2+y^2$ has vertex at (0,0,0), we expect $8-x^2-y^2\geq x^2+y^2$, for $(x,y)\in D$, but we should check this. One way is to see that the difference $(8-x^2-y^2)-(x^2+y^2) = 8-2(x^2+y^2)\geq 0$, for all $(x,y)\in D$. Since D is closed and bounded, there is an absolute maximum and absolute minimum value for $h(x,y) = 8 - 2(x^2+y^2)$ over D. We want this absolute minimum value to be greater than or equal to zero. If we set $h_x = 0 = h_y$, we see that (0,0) is a critical point in the interior of D with value $h(0,0) = 8$. Note, for our present purpose, it does not matter if 8 is a relative maximum or minimum. The boundary of the square D consists of four edges. We won't check each of these, but along the right hand edge of D, we have $x = 1$ and $-1 \leq y \leq 1$. Thus, $h(1,y) = 6 - 2y^2$. The smallest value of $h(1,y)$, for $-1 \leq y \leq 1$ is 4, since 1 is the largest value of y^2. In a similar way, one can see that $h(x,y) \geq 0$ for all $(x,y) \in D$, so $f(x,y) \geq g(x,y)$, for all $(x,y) \in D$. (In fact, the absolute minimum value of $h(x,y)$ over D is 4, and the absolute maximum value is 8.) Thus, $\int_{-1}^{1}\int_{-1}^{1} 8 - 2(x^2+y^2)\ dy\ dx$ gives the required volume. This double integral is easily seen to be $\frac{80}{3}$.

Example 2. Find the volume of the region in $\mathbb{R}^3$ enclosed by the graphs of $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$.



.

Solution: Note that in this version of the example, we are not given a domain of integration. To find D, the domain of integration, we set $8 - x^2 - y^2 = x^2 + y^2$, to see that the paraboloids intersect along the circle $x^2 + y^2 = 4$, elevated to the level $z = 4$. Since the functions are continuous and the first dominates the second at the origin, the first will dominate the second for all $(x,y) \in D$, where $D : 0 \leq x^2 + y^2 \leq 4$. Thus, the volume in questions is:

$$\begin{aligned}
\int\int_D (8 - x^2 - y^2) - (x^2+y^2)\ dA &= \int\int_D 8 - 2(x^2+y^2)\ dA \\
&= \int_0^{2\pi}\int_0^2 \{8 - 2((r\cos(\theta))^2 + (r\sin(\theta))^2)\} r\ dr\ d\theta \\
&= \int_0^{2\pi}\int_0^2 8r - 2r^3\ dr\ d\theta \\
&= \int_0^{2\pi} (4r^2 - \frac{1}{2}r^4)\Big|_{r=0}^{r=2}\ d\theta \\
&= \int_0^{2\pi} (16-8) - 0\ d\theta \\
&= \int_0^{2\pi} 8\ d\theta \\
&= 16\pi.
\end{aligned}$$

Wednesday, March 17. We discussed improper double integrals. Recall from Calculus 2 an improper integral can be either of the form: $\int_a^b f(x)\ dx$, where $f(x)$ is unbounded on $[a,b]$ or $\int_a^\infty f(x)\ dx$, where the domain of integration is unbounded. Even if $f(x)$ is continuous, the integrals above may not exist (converge).

Example 1. $\int_0^1 \frac{1}{\sqrt{x}}\ dx$. Note that $f(x)$ is unbounded on $[0,1]$, but $\lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\ dx = 2$ exists, so the original integral exists.

Example 2. $\int_1^\infty e^{-2x}\ dx$. Note that the domain of integration is unbounded, but $\lim_{b\to\infty}\int_1^b e^{-2x}\ dx = \frac{1}{2e^2}$ exists, so the original integral exists.

Similar approaches for integrals like $\int_{-1}^{1}\frac{1}{\sqrt[3]{x}}\ dx$ and $\int_{-\infty}^{\infty} e^{-x}\ dx$ apply. Of course, improper single integrals need not converge, e.g., $\int_0^1 \frac{1}{x^2}\ dx$. Here is an example of an improper double integral:

Example 3. $\int\int_D \frac{1}{\sqrt{1-x^2-y^2}}\ dA$, where $D : 0 \leq x^2+y^2 \leq 1$. Note that the integrand $f(x,y) = \frac{1}{\sqrt{1-x^2-y^2}}$ approaches infinity as points in the interior of the disk approach the circle $x^2+y^2=1$. Thus the integrand is unbounded on the domain of integration. Let D_a denote the disk $0 \leq x^2+y^2 \leq a^2$, with $0 < a < 1$. If the $\lim_{a\to 1}\int\int_{D_\epsilon} f(x,y)\ dA$ exists, then it will converge to the original integral, i.e.,

$$\int\int_D \frac{1}{\sqrt{1-x^2-y^2}}\ dA = \lim_{a\to 1}\int\int_{D_a}\frac{1}{\sqrt{1-x^2-y^2}}\ dA.$$

We can use polar coordinates:

$$\begin{aligned}\int\int_{D_a}\frac{1}{\sqrt{-1-x^2-y^2}}\ dA &= \int_0^{2\pi}\int_0^a \frac{r}{\sqrt{1-r^2}}\ drd\theta \\ &= \int_0^{2\pi} -\sqrt{1-r^2}\Big|_{r=0}^{r=a}\ dA \\ &= \int_0^{2\pi} -\sqrt{1-a^2}+1\ dr \\ &= 2\pi\cdot(-\sqrt{1-a^2}+1).\end{aligned}$$

Taking the limit as $a \to 1$, we get 2π. Thus:

$$\int\int_D \frac{1}{\sqrt{1-x^2-y^2}}\ dA = 2\pi.$$

Example 4. $\int\int_D xye^{-x^2-y^2}\ dA$, where D is the first quadrant of $\mathbb{R}^2$.

Solution: In this case we can proceed as one might expect:

$$\int\int_D xye^{-x^2-y^2}\ dA = \lim_{a,b\to\infty}\int_0^b\int_0^a xye^{-x^2-y^2}\ dxdy.$$

Here's one way to evaluate the iterated integral:

$$\begin{aligned}\int_0^b\int_0^a xye^{-x^2-y^2}\ dxdy &= \int_0^b\int_0^a (xe^{-x^2})(ye^{-y^2})\ dxdy \\ &= \int_0^b ye^{-y^2}(\int_0^a xe^{-x^2}\ dx)\ dy \\ &= (\int_0^b ye^{-y^2}\ dy)\cdot(\int_0^a xe^{-x^2}\ dx).\end{aligned}$$

Calculating these integrals separately:

$$\begin{aligned}\int_0^b ye^{-y^2}\ dy &= -\frac{1}{2}e^{-y^2}\Big|_{y=0}^{y=b} \\ &= \frac{1}{2}(-e^{-b^2}+1).\end{aligned}$$

Similarly: $\int_0^a xe^{-x^2}\ dx = \frac{1}{2}(-e^{-a^2}+1)$. Thus:

$$\int_0^b\int_0^a xye^{-x^2-y^2}\ dxdy = \frac{1}{2}(-e^{-b^2}+1)\cdot\frac{1}{2}(-e^{-a^2}+1).$$

Passing to the limit at $a \to \infty$ and $b \to \infty$, we get:

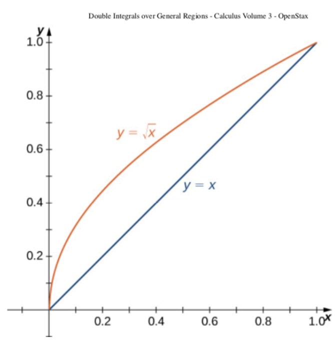
$$\int\int_D xye^{-x^2-y^2}\ dA = \frac{1}{4}.$$

There are many versions of Fubini's Theorem for improper integrals. Here is one:

Theorem (Fubini). If D is a bounded region in the plane defined by $a \le x \le b$ and $c(x) \le y \le d(x)$ or $c \le y \le d$ and $a(y) \le x \le b(y)$, and $f(x, y)$ is a continuous non-negative function on D which has finitely many discontinuities in the interior of D, then:

$$\begin{aligned}\int\int_D f(x,y)\ dA &= \int_a^b \int_{c(x)}^{d(x)} f(x,y)\ dydx \\ &= \int_c^d \int_{a(y)}^{b(y)} f(x,y)\ dxdy.\end{aligned}$$

Example 5. $\int\int_D \frac{e^y}{y}\ dA$, where D is:



Solution: Note there are no discontinuities in the interior of D, so we can use Fubini's Theorem. Integrating with respect to x first, we have

$$\begin{aligned}\int\int_D \frac{e^y}{y} dA &= \int_0^1 \int_{y^2}^{y} \frac{e^y}{y}\ dxdy \\ &= \int_0^1 \frac{e^y}{y} x\Big|_{x=y}^{x=y^2}\ dy \\ &= \int_0^1 \frac{e^y}{y} \cdot (y^2 - y)\ dy \\ &= \int_0^1 ye^y - e^y\ dy \\ &= (ye^y - 2e^y)\Big|_{y=0}^{y=1} \\ &= (e - 2e) - (0 - 2) \\ &= 2 - e.\end{aligned}$$

Example 6. Consider $\int\int_D \frac{x^2+1}{\sqrt[3]{y-1}}\ dA$, where D is the rectangle $0 \le x \le 1$ and $0 \le y \le 2$. Note that the integrand $f(x, y) = \frac{x^2+1}{\sqrt[3]{y-1}}$ is discontinuous along the line $y = 1$, which accounts for infinitely many discontinuities in the interior of D. But we can write $D = D_1 \cup D_2$, where $D_1 : 0 \le x \le 1, 0 \le y \le 1$ and $D_2 : 0 \le x \le 1, 1 \le y \le 2$. Fubini's Theorem applies to both integrals $\int\int_{D_1} f(x, y)\ dA$ and $\int\int_{D_2} f(x, y)\ dA$ and we then have:

$$\int\int_D f(x,y)\ dA = \int\int_{D_1} f(x,y)\ dA + \int\int_{D_2} f(x,y)\ dA.$$

Both integrals on the right above reduce to an improper single integral.

For example:

$$\int\int_{D_1} f(x,y)\ dA = \int_0^1 \int_0^1 \frac{x^2+1}{\sqrt[3]{y-1}}\ dxdy$$
$$= \int_0^1 \frac{1}{\sqrt[3]{y-1}}(\frac{x^3}{3}+x)\Big|_{x=0}^{x=1}\ dy$$
$$= \int_0^1 \frac{4}{3\sqrt[3]{y-1}}\ dy,$$

which is an improper integral. Working this out in the usual way, one gets -2. The double integral $\int\int_{D_1} f(x,y)\ dA$ can be worked in a similar fashion.

Example 7. This example shows how things can go wrong in Fubini's Theorem if $f(x,y)$ is not non-negative on the domain of integration. Consider: $\int\int_D f(x,y)\ dA$, for $D: 0 \le x \le 2, 0 \le y \le 1$ and

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{(x^2+y^2)^3}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}.$$

Integrating with respect to y first, we have

$$\int\int_D f(x,y)\ dA = \int_0^2 \int_0^1 \frac{xy(x^2-y^2)}{(x^2+y^2)^3}\ dydx.$$

Noting that x is a constant in the inner integral we substitute $u = x^2 + y^2$, with $du = 2ydy$. We then get:

$$\int_0^2 \int_0^1 \frac{xy(x^2-y^2)}{(x^2+y^2)^3}\ dydx = \int_0^2 \int_{x^2}^{x^2+1} \frac{x(2x^2-u)}{2u^3}\ dudx$$
$$= \int_0^2 \int_{x^2}^{x^2+1} (\frac{x^3}{u^3} - \frac{x}{2u^2})\ dudx$$
$$= \int_0^2 (-\frac{x^3}{2u^2} + \frac{x}{2u})\Big|_{u=x^2}^{u=x^2+1}\ dx$$
$$= \int_0^2 -\frac{x^3}{2(x^2+1)^2} + \frac{x}{2(x^2+1)} + \frac{1}{2x} - \frac{1}{2x}\ dx$$
$$= \int_0^2 \frac{x}{2(x^2+1)^2}\ dx$$
$$= -\frac{1}{4(x^2+1)}\Big|_0^2$$
$$= -\frac{1}{20} + \frac{1}{4} = \frac{1}{5}.$$

A similar calculation shows that if we switch the order of integration:

$$\int\int_D \frac{xy(x^2-y^2)}{(x^2+y^2)^3}\ dA = \int_0^1 \int_0^2 \frac{xy(x^2-y^2)}{(x^2+y^2)^3}\ dxdy$$
$$= -\frac{1}{20}.$$

What goes wrong here? It turns out that the discontinuity at (0,0) is too severe for $\int\int_D f(x,y)\ dA$ to exist over the given domain, even though the individual iterated integrals exist. This is because $f(x,y)$ quickly approaches $+\infty$ or $-\infty$ depending on the approach to (0,0), so that the Riemann sums defining the double integral do not converge.

Here is another version of Fubini's Theorem for improper double integrals.

Theorem (Fubini). Suppose $f(x, y)$ is continuous on $R = [a, \infty] \times [b, \infty]$. If the limits exist, then:

$$\begin{aligned}\int\int_R f(x,y)\ dA &= \lim_{(b,d)\to\infty} \int_a^b \int_c^d f(x,y)\ dydx \\ &= \lim_{(b,d)\to\infty} \int_c^d \int_a^b f(x,y)\ dxdy.\end{aligned}$$

and if $a = b = 0$,

$$\int\int_D f(x,y)\ dA = \lim_{n\to\infty} \int_0^{\frac{\pi}{2}} \int_0^n f(r\cos(\theta), r\sin(\theta))\ rdrd\theta$$

Here is a classical application of the preceding theorem.

Example 8. To calculate $\int_0^\infty e^{-x^2}\ dx$, we calculate $(\int_0^\infty e^{-x^2}\ dx)^2$.

Solution:

$$\begin{aligned}(\int_0^\infty e^{-x^2}\ dx)^2 &= (\lim_{a\to\infty} \int_0^a e^{-x^2}\ dx)^2 \\ &= \lim_{a\to\infty} (\int_0^a e^{-x^2}\ dx) \cdot (\int_0^a e^{-y^2}\ dy) \\ &= \lim_{a\to\infty} \int_0^a \int_0^a e^{-x^2-y^2}\ dydx \\ &= \lim_{n\to\infty} \int_0^{\frac{\pi}{2}} \int_0^n e^{-r^2} r\ drd\theta. \\ &= \lim_{n\to\infty} \int_0^{\frac{\pi}{2}} -\frac{1}{2}e^{-r^2}\Big|_0^n\ d\theta \\ &= \lim_{n\to\infty} \int_0^{\frac{\pi}{2}} -\frac{1}{2}e^{-n^2} + \frac{1}{2}\ d\theta \\ &= \lim_{n\to\infty} (-\frac{1}{2}e^{-n^2} + \frac{1}{2}) \cdot \frac{\pi}{2} \\ &= \lim_{n\to\infty} \frac{\pi}{4}(-e^{-n^2} + 1) \\ &= \frac{\pi}{4}.\end{aligned}$$

Taking the square root of both sides, we have:

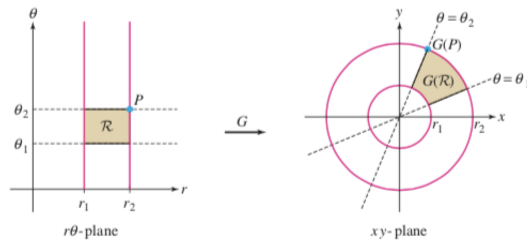
$$\int_0^\infty e^{-x^2}\ dx = \frac{\sqrt{\pi}}{2}.$$

By symmetry it follows that:

$$\int_{-\infty}^\infty e^{-x^2}\ dx = \sqrt{\pi},$$

a fundamental formula concerning normal distributions in probability theory.

Thursday, March 18. We began our discussion of the change of variables principle for double integrals. We noted that one of the purposes of this principle is that it transforms a double integral over a domain of integration that may be difficult to integrate over into a double integral over a domain of integration that is more manageable. The example of this we have already seen is the use of polar coordinates. We can think of using polar coordinates as changing variables from x and y to r and θ. If we write $G(r, \theta) = (r\cos(\theta), r\sin(\theta))$, then we can think of G as a function that transforms verticle lines in the (r,θ)-plane to arcs on circles in the xy-plane .

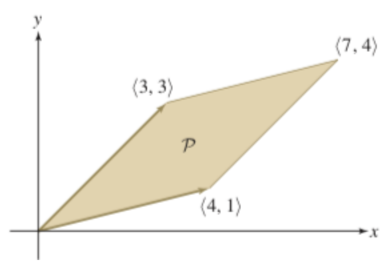


Note that $G(r,\theta)$ takes any vertical line $r = r_0$ in the r,θ-plane and wraps it infinitely many times around the circle of radius r_0 centered at the origin in the xy-plane. If $r = r_0$ and $0 \le \theta < 2\pi$, then G applied to this vertical line seqment in the r,θ-plane is the circle of radius r_0 (no points repeated) centered at (0,0) in the xy-plane. G also takes the rectangle $\mathcal{R}$ in the uv-plane, in the diagram above, to the polar rectangle $G(\mathcal{R})$ in the xy plane.

The r in the equation $dA = r\ dr\ d\theta$ comes from the *Jacobian* of the polar transformation

$$\text{Jac}(G) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix} = r\cos^2(\theta) + r\sin^2(\theta) = r.$$

Example 1. Consider $\int\int_{\mathcal{P}} 3x + 2y\ dA$ for $\mathcal{P}$ the region:



A close look at $\mathcal{P}$ shows that if we try to think of $\mathcal{P}$ as a region of Type 1 or Type 2, we will have to subdivide $\mathcal{P}$ into three parts.

However, we can change variables.

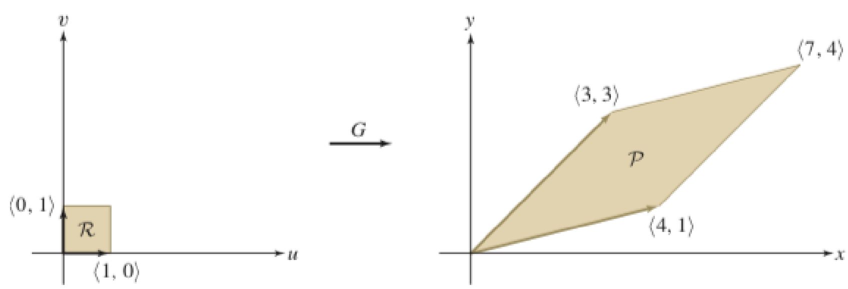
Set $x = 4u + 3v, y = u + 3v$, or equivalently, define $G(u,v) = (4u + 3v, u + 3v)$. We take the absolute value of the determinant of the 2×2 matrix of partial derivatives:

$$\det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det \begin{pmatrix} 4 & 3 \\ 1 & 3 \end{pmatrix} = 9 = |9|$$

and set $dA = 9\ du\ dv$. Now we substitute:

$$\begin{aligned}\int\int_{\mathcal{P}} 3x + 2y\ dA &= \int_0^1 \int_0^1 3(4u + 3v) + 2(u + 3v)\ 9dudv \\ &= 9\int_0^1 \int_0^1 14u + 15v\ dudv \\ &= 9\int_0^1 (7u^2 + 15uv)\Big|_{u=0}^{u=1} dv \\ &= 9\int_0^1 7 + 15v\ dv \\ &= 9(7v + \frac{15}{2}v^2)\Big|_0^1 \\ &= 9(7 + \frac{15}{2}) \\ &= \frac{261}{2}.\end{aligned}$$

Where does this come from? We have a transformation (function) $G : \mathbb{R}^2 \to \mathbb{R}^2$ which takes (u, v) in the uv-plane to $(4u + 3v, u + 3v)$ in the xy-plane:



Let's see how G transforms $\mathcal{R}$ to $\mathcal{P}$. Note that $\mathcal{P}$ is the parallelogram spanned by the vectors (4,1) and (3,3). $G(0, 0) = (0, 0)$, $G(0, 1) = (3, 3)$,G(1,0) = (4,1), and $G(1, 1) = (7, 4)$ showing that G takes the corners of the unit square in the uv-plane to the corners of the parallelogram $\mathcal{P}$ in the xy-plane. .

We can also verify that G takes any point on the u-axis in the uv-plane to a point on the line through (0,0) and $(4, 1)$. For example, $G(a, 0) = (4a, a)$, which lies on the line $y = \frac{1}{4}x$. Note that if $0 \le a \le 1$, then $(4a, a)$ lies on the line segment through (0,0) and (4,1). Thus, G transforms the lower edge of $\mathcal{R}$ to the line segment in $\mathcal{P}$ connecting $(0, 0)$ and $(4, 1)$.

Since $G(0, 1) = (3, 3)$ and $G(1, 1) = (7, 4)$, in a similar way one can see that G transforms each of the edges of $\mathcal{R}$ into corresponding edges of $\mathcal{P}$.

Finally, if (a, b) is in the interior of $\mathcal{R}$, then $0 < a < 1$ and $0 < b < 1$. The slope of the line through (0,0) in the xy-plane and $G(a, b)$ is $\frac{1}{4} \le \frac{u+3v}{4u+3v} \le 1$, which shows that $G(a, b)$ lies in the interior of $\mathcal{P}$.

Thus, G transforms $\mathcal{R}$ into $\mathcal{P}$.

Definition. (i) A transformation is a function $G(u, v) = (x(u, v), y(u, v))$, from the uv-plane to the xy-plane, The *Jacobian* of $G(u, v)$ is the function

$$\mathrm{Jac}(G) := \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}.$$

We will also write $\mathrm{Jac}(G) = \frac{\partial(x,y)}{\partial(u,v)}$. We will assume that our transformations satisfy the property that *all first order partial derivatives* $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$ *exist and are continuous* in the domain of $G(u, v)$.

(ii) The transformation $G(u,v)$ is said to be *one-to-one* if no two points in the uv-plane go to the same point in the xy-plane under the transformation $G(u,v)$. i,e., $G(u_1,v_1) = G(u_2,v_2)$ implies $(u_1,v_1) = (u_2,v_2)$.

The Example 1 above is a special case of a very important type of transformation:

Linear Transformations. A transformation T from the uv-plane to the xy-plane is said to be a *linear transformation* if $T(u,v) = (au+bv, cu+dv)$, for constants $a,b,c,d \in \mathbb{R}$. Note that in this case we have

$$\text{Jac}(T) = \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Important Properties of linear transformations.

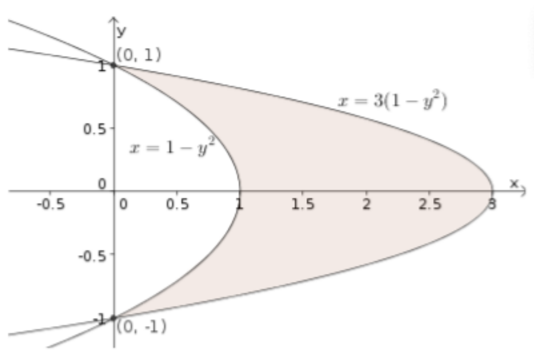
(i) $T(P_1+P_2) = T(P_1)+T(P_2)$, for points P_1, P_2 in the uv-plane.
(ii) $T(\lambda P) = \lambda T(P)$, for all points P in the uv-plane and $\lambda \in \mathbb{R}$.
(iii) T is one-to-one if and only if the Jacobian $ad - bc \neq 0$

Part (ii) shows that T takes lines through the origin in the uv-plane to lines through the origin in the xy-plane. In general, T transforms the unit square in the uv-plane to the parallelogram in the xy-plane spanned by the vectors (a,c) and (b,d). You can see this using the same reasoning as in our first example above. This tells us how to find a transformation T that takes the unit square in the uv-plane to a given parallelogram in the xy-plane. For example, if P is the parallelogram in the xy-plane with corners $(0,0), (-1,4), (2,3), (1,7)$, then we see that $T(u,v) = (-u+2v, 4u+3v)$ transforms the unit square in the uv-plane to P.

Friday, March 19. We continued our discussion of the change of variables principle for double integrals, first by reviewing the definition of a transformation of the uv-plane to the xy-plane and its Jacobian, and then reviewing the special case of a linear transformation. We then gave more examples of transformations.

Example 1. Consider the transformation $G(u,v) = (v(1-u^2), u)$. $\text{Jac}(G) = \det\begin{pmatrix} -2uv & 1-u^2 \\ 1 & 0 \end{pmatrix} = -(1-u^2)$.

We show G takes the rectangle $-1 \leq u \leq 1$, $1 \leq v \leq 3$ in the uv-plane to the region below in the xy-plane.



If we fix $1 \leq v_0 \leq 3$, and let u vary, $G(u,v_0) = (v_0(1-u^2), u)$, so $x = v_0(1-y^2)$ gives a parabola between the two indicated parabolic boundaries, with the one on the far right occurring when $v_0 = 3$ and the one on the right occurring when $v_0 = 1$. Note that for each v_0, as u varies from -1 to 1, we just get that portion of the corresponding parabola lying in the shaded region.

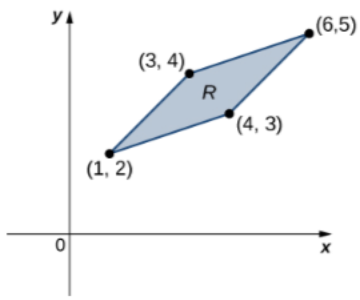
Example 2. Translation. Let $T(u,v) = (u+a, v+b)$. Then this is the transformation obtained by translating the origin of the uv-plane to the point (a,b) in the xy-plane.

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

For example: If $u^2+v^2 = R^2$, and $T(u,v) = (u+a, v+b)$, then $u = x-a$ and $v = y-b$. It follows that $(x-a)^2 + (y-b)^2 = R^2$.

In other words, T translates the circle (and the disk) of radius R in the uv-plane centered at (0,0) to the circle (and disk) of radius R in the xy-plane, centered at (a, b).

Example 3. We can combine different types of transformations to create new ones. For example, let's determine the transformation which takes the unit square in the uv-plane to the parallelogram



in the xy-plane. We see that this parallelogram is the translation of one similar to it centered at the origin.

If we move the vertex (1,2) to the origin,we get a new parallelogram with vertices (0,0), (2,2), (5,3), (3,1), moving counterclockwise along the perimeter. This new parallelogram is spanned by the vectors (3,1) and (2,2), so that by the previous lecture, $T(u, v) = (3u + 2v, u + 2v)$ takes the unit square in the uv-plane to the new parallelogram. If we add (1,2) to the new coordinates, we get

$$G(u, v) = (3u + 2v + 1, u + 2v + 2),$$

and this transformation takes the unit square in the uv-plane to the original parallelogram in the xy-plane.

Notice that G takes the vertices (0,0), (1,0), (1,1), (0,1), in the uv-plane to the vertices (1,2), (4,3), (6,5), (3,4) in the xy-plane. Moreover, $\text{Jac}(G) = \det \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} = 4$.

Here is the theorem that tells us how to change variables in a double integral.

Theorem. Let $G(u, v) = (x(u, v), y(u, v))$ be a transformation from the uv-plane to the xy-plane. Suppose D_0 is a subset of the uv-plane and write $D = G(D_0)$. Assume $G(u, v)$ is one-to-one on the interior of D_0. Then:

$$\begin{aligned} \int\int_D f(x, y)\ dA &= \int\int_{D_0} f(x(u, v), y(u, v))\ |Jac(G)|\ dA \\ &= \int\int_{D_0} f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dA, \end{aligned}$$

where $|\text{Jac}(G)| = |\frac{\partial(x,y)}{\partial(u,v)}|$ denotes the absolute value of the Jacobian of G. The crucial point in this formula is that small portions of area dA in the xy-plane become $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ times small portions of area dA in the uv-plane. In particular, if D_0 is a region of Type 1 in the uv-plane, $D_0 : a \le u \le b$ and $c(u) \le v \le d(u)$, then we have

$$\int\int_D f(x, y)\ dA = \int_a^b \int_{c(u)}^{d(u)} f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dv\ du,$$

and a similar formula holds if D_0 is a region of Type 2.

Example 4. Calculate $\int\int_D (x + y)\ dA$, where D is the region give in Example 3.

Solution: $G(u,v) = (3u + 2v + 1, u + 2v + 2)$, with D_0 the unit square in the uv-plane. We have $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = |4| = 4$. Since G is one-to-one (check this), we have

$$\int\int_D (x+y)\ dA = \int_0^1 \int_0^1 \{(3u + 2v + 1) + (u + 2v + 2)\} \cdot 4dudv.$$
$$= \int_0^1 \int_0^1 16u + 16v + 12\ dudv$$
$$= \int_0^1 (8u^2 + 16uv + 12u)\Big|_{u=0}^{u=1} dv$$
$$= \int_0^1 16v + 20\ dv$$
$$= (8v^2 + 20v)\Big|_0^1$$
$$= 28.$$

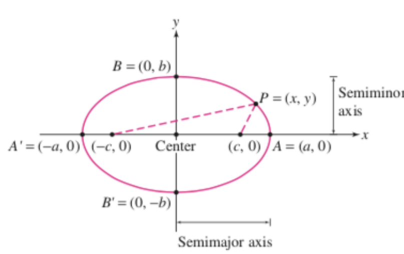
Example 5. $\int\int_D \frac{y^2}{x}\ dA$, where D is the region in Example 1.

Solution: We have $G(u,v) = (v(1-u^2), u)$ with D_0 the rectangle $-1 \le u \le 1$, $1 \le v \le 3$ in the uv-plane. And: $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = |-(1-u^2)|$. Since $-1 \le u \le 1$, we have $|-(1-u^2)| = 1 - u^2$. In other words, $1 - u^2$ is greater than or equal to zero for $-1 \le u \le 1$. We can also see that G is one-to-one on the interior of D_0, therefore,

$$\int\int_D \frac{y^2}{x}\ dA = \int_1^3 \int_{-1}^1 \frac{u^2}{v(1-u^2)} \cdot (1-u^2)\ dudv$$
$$= \int_1^3 \int_{-1}^1 \frac{u^2}{v}\ dudv$$
$$= \int_1^3 \frac{u^3}{3v}\Big|_{u=-1}^{u=1} dv$$
$$= \frac{2}{3}\int_1^3 \frac{1}{v}\ dv$$
$$= \frac{2}{3}\ln 3.$$

Example 6. Calculate the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution: For this, we will use the fact that the area of the ellipse is area$(D) = \int\int_D dA$, where D is the region in the xy-plane given by $D: 0 \le \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$. We use the transformation $G(u,v) = (au\cos(v), bu\sin(v))$, with $(u,v) \in D_0 = [0,1] \times [0,2\pi]$, to calculate $\int\int_D\ dA$.



Notice that if we set $u = 1$, we get $G(1,v) = (a\cos(v), b\sin(v))$, and since $\frac{a\cos(v))^2}{a^2} + \frac{(b\sin(v))^2}{b^2} = 1$, this shows that G takes the right hand edge of the rectangle D_0 to the given ellipse. For any $0 < u_0 < 1$ $G(u_0, v)$ is a smaller ellipse, and these together fill the interior of the solid ellipse D. $G(0,v)=(0,0)$, which gives the origin. Thus, $G(D_0)=D$. We also have $\frac{\partial(x,y)}{\partial(u,v)}=\det\begin{pmatrix} a\cos(v) & -au\sin(v) \\ b\sin(v) & bu\cos(v)\end{pmatrix}=abu\cos^2(v)+abu\sin^2(v)=abu$. Therefore: $|\frac{\partial(x,y)}{\partial(u,v)}|=|abu|=abu$, since $0\le u\le 1$. We note that it follows from the non-vanishing of the Jacobian on the interior of D_0 that G is one-to-one on the interior of D_0. Thus, we may apply the change of variables theorem.

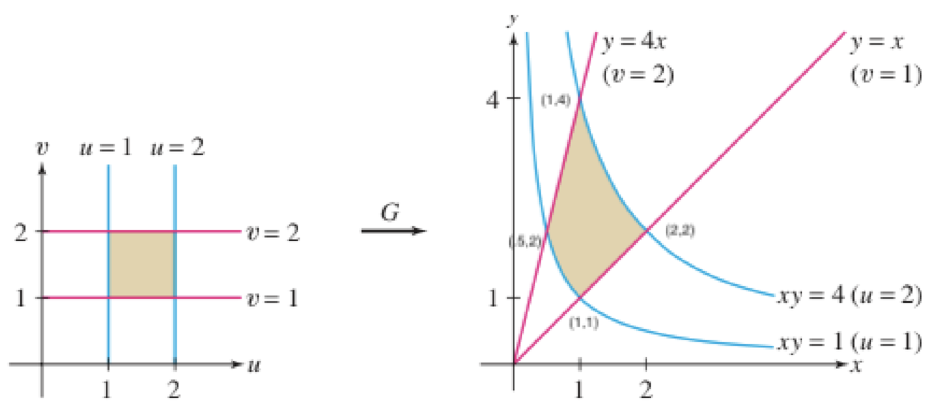
Thus, the area of the ellipse is

$$\begin{aligned}\int\int_D dA &= \int\int_{D_0}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA\\ &= \int_0^{2\pi}\int_0^1 abu\ dudv\\ &= ab\int_0^{2\pi}\frac{u^2}{2}\Big|_0^1\ dv\\ &= ab\int_0^{2\pi}\frac{1}{2}\ dv\\ &= \pi ab.\end{aligned}$$

Monday, March 22. We continued our discussion concerning change of variable for double integrals.

Example 1. Calculate $\int\int_D xy\ dA$ for $D\subseteq\mathbb{R}^2$ bounded by $xy=4$, $xy=1$, $y=4x$ and $y=x$.

Solution: We work with the change of variables $G(u,v)=(uv^{-1},uv)$ which transforms the rectangle R in the uv-plane below to the region D.

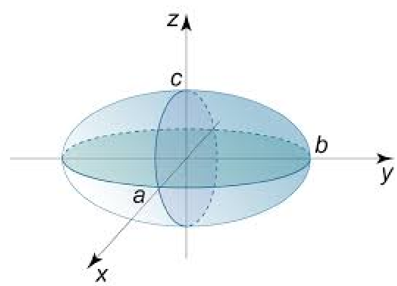


To see that G transforms R into D, notice that G applied to the line $v=1$ yields $G(u,1)=(u,u)$ which is the line $y=x$, while $G(u,2)=(\frac{u}{2},2u)$, which is the line $y=4x$. Moreover, $G(1,v)=(\frac{1}{v},v)$ which is the hyperbola $xy=1$ and $G(2,v)=(\frac{2}{v},2v)$, which is the hyperbola $xy=4$. Moreover, we can easily check that $G(u,v)$ takes corners of R to corners of D. For example, $G(1,1)=(1,1)$, which is where the hyperbole $xy=1$ intersects the line $y=x$. $G(1,2)=(\frac{1}{2},2)$ which is where the hyperbole $xy=1$ intersects the line $y=4x$. Similarly for the other corners of D and R. This shows that, in fact, G does transform R into D. The question of how we can find R (rather than checking that a given R works) can be answered in terms of inverses, which we will see below.

For the Jacobian of $G(u,v)$, we have $\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} v^{-1} & -uv^{-2} \\ v & u \end{pmatrix} = 2uv^{-1}$. Since $1 \leq u, v \leq 2$, $2uv^{-1}$ is positive on R, so $|\frac{\partial(x,y)}{\partial(u,v)}| = 2uv^{-1}$. Thus, we have

$$\begin{aligned}
\int\int_D xy\ dA &= \int_1^2 \int_1^2 (uv^{-1})(uv) \cdot 2uv^{-1}\ du\ dv \\
&= \int_1^2 \int_1^2 2u^3v^{-1}\ dudv \\
&= \frac{1}{2}\int_1^2 u^4v^{-1}\Big|_{u=1}^{u=2}\ dv \\
&= \frac{15}{2}\int_1^2 v^{-1}\ dv \\
&= \frac{15}{2}(\ln(2) - \ln(1)) = \frac{15}{2}\ln(2).
\end{aligned}$$

Example 2. Calculate the volume of the ellipsoid E: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.



Solution: If we let D be the elliptic disk $0 \leq \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$, then:

$$\text{vol}(E) = 2\int\int_D c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\ dA.$$

We can do this two different ways using the change of variables formula. First, we can use the transformation from the end of last lecture, $G(u,v) = (au\cos(v), bu\sin(v))$, with $|\frac{\partial(x,y)}{\partial(u,v)}| = abu$, which takes the rectangle $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$ to D. We then get:

$$\begin{aligned}
\text{vol}(E) &= 2\int_0^{2\pi}\int_0^1 c\sqrt{1 - \frac{(au\cos(\theta))^2}{a^2} - \frac{(bu\sin(\theta))^2}{b^2}}\ abu\ dudv \\
&= 2abc\int_0^{2\pi}\int_0^1 u\sqrt{1-u^2}\ dudv \\
&= 2\int_0^{2\pi} -\frac{1}{3}(1-u^2)^{\frac{3}{2}}\Big|_{u=0}^{u=1}\ dv \\
&= 2abc\int_0^{2\pi}\frac{1}{3}\ dv \\
&= \frac{4}{3}\pi abc
\end{aligned}$$

Alternately, we can use the linear transformation $T(u,v) = (au, bv)$, with $|\frac{\partial(x,y)}{\partial(u,v)}| = ab$. This transformation stretches the plane a units horizontally and b units vertically. It takes the unit disk $D' : 0 \leq u^2 + v^2 \leq 1$ to

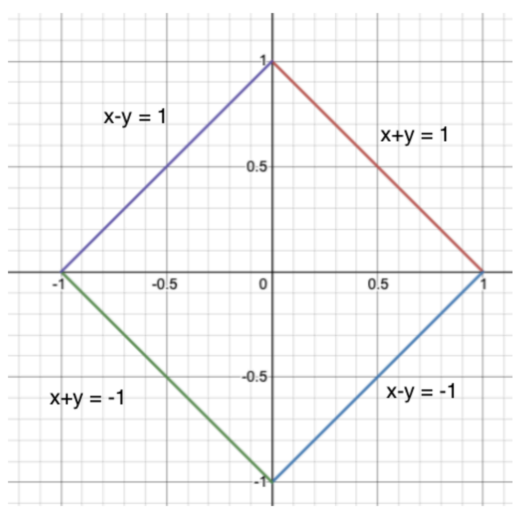
D. Thus:

$$\begin{aligned}\text{vol}(E) &= 2\int\int_D c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\ dA.\\ &= 2c\int\int_{D'}\sqrt{1-\frac{(au)^2}{a^2}-\frac{(bv)^2}{b^2}}ab\ dudv\\ &= 2abc\int\int_{D'}\sqrt{1-u^2-v^2}\ dudv.\end{aligned}$$

Now, we can either use polar coordinates to evaluate the last double integral, or in this case, recognize it as the volume of the top half of the unit sphere, which is $\frac{2}{3}\pi$. Thus, $\text{vol}(E) = \frac{4}{3}\pi abc$.

The next example shows that a change of variables can be used when the integrand has no (obvious) antiderivative.

Example 3. Calculate $\int\int_D (x+y)^2 e^{x^2-y^2}\ dA$, where D is the diamond with vertices (1,0), (-1,0), (0,1), (0,-1).



Solution: Note that as written, the integrand does not have an antiderivative with respect to either variable. If we could find a transformation $G(u,v)$ having the property that $x+y=u$ and $x-y=v$, then the integrand becomes u^2e^{uv}, which is manageable. This suggests that to find $G(u,v)$, we must solve for x and y in terms of u and v. Adding the two equations gives $2x = u+v$, so $x = \frac{u+v}{2}$. Subtracting gives $2y = u-v$, so $y = \frac{u-v}{2}$. Therefore, we take $G(u,v) = (\frac{u+v)}{2}, \frac{u-v}{2})$ is linear so the pre-image D_0 of D must at least be a parallelogram. The corners of D are (1,0), (-1,0), (0,1), (0,-1). Substituting these points into the equations for u and v gives, (1,1), (-1,-1), (1,-1), (-1,1). Thus, D_0 is the rectangle $[-1,1]\times[-1,1]$ in the uv-plane. In other words, $G(u,v)$ transforms D_0 into D.

We also have $\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = -\frac{1}{2}$. Thus, $|\text{Jac}(G)| = \frac{1}{2}$. Therefore,

$$\begin{aligned} \int\int_D (x+y)^2 e^{x^2-y^2}\ dA &= \int\int_{D_0} u^2 e^{uv}\ \frac{1}{2} du dv \\ &= \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} u^2 e^{uv}\ dv du \\ &= \frac{1}{2}\int_{-1}^{1} ue^{uv}\Big|_{v=-1}^{v=1}\ du \\ &= \frac{1}{2}\int_{-1}^{1} u(e^u - e^{-u})\ du \\ &= 2e^{-1}, \end{aligned}$$

the last step being a standard Calculus 2 problem that can be solved using integration by parts.

What is really happening in the previous example is the following. We knew how to express u, v in terms of x, y, but really wanted to express x, y in terms of u, v. What this means, is that we were given a function $F(x, y) = (u(x, y), v(x, y))$ that writes u, v in terms of x, y, and we want to "unravel" $F(x, y)$ so that we can express x, y in terms of u, v, via the function $G(u, v)$. This involves realizing $G(u, v)$ as the inverse of $F(x, y)$, or equivalently, regarding $F(x, y)$ as the inverse of $G(u, v)$. We will talk about inverse transformations in the next lecture.

Tuesday, March 23. We discussed the notion of the inverse of our change of variables transformation $G(u, v)$ and how it can help determine what the domain of integration is, after changing coordinates. If we think of $G(u, v) = (x(u, v), y(u, v))$ as a transformation of the uv-plane to the xy-plane, then a transformation $F(x, y) = (u(x, y), v(x, y))$ taking points in the xy-plane to points in the uv-plane is the *inverse of G(u,v)* if $F(G(u, v) = (u, v)$ for all (u, v) in the domain of G and $G(F(x, y)) = (x, y)$ for all (x, y) in the domain of F. It is important that both these equations hold, otherwise F is not the inverse of G. By symmetry, it follows that G is the inverse of F. While it may not always be possible to find F given G, the idea is that if we express x and y in terms of u and v, we try to solve for u and v in terms of x and y to find F. Conversely, if we are given equations expressing u and v in terms of x and y, we regard this as F, and if we can solve for x and y in terms of u and v this gives G.

Example 1. Suppose $G(u, v) = (3u + 4v, 2u + 3v)$ and $F(x, y) = (3x - 4y, -2x + 3y)$, then

$$\begin{aligned} F(G(u, v)) &= F(3u + 4v, 2u + 3v) \\ &= (3(3u + 4v) - 4(2u + 3v), -2(3u + 4v) + 3(2u + 3v)) \\ &= (u, v). \end{aligned}$$

On the other hand,

$$\begin{aligned} G(F(x, y)) &= G(3x - 4y, -2x + 3y)) \\ &= (3(3x - 4y) + 4(-2x + 3y), 2(3x - 4y) + 3(-2x + 3y)) \\ &= (x, y), \end{aligned}$$

showing that F is the inverse of G. Note that if we are given $G(u, v)$, so that $x = 3u + 4v$ and $y = -2x + 3y$, we will get $F(x, y)$ by solving for u and v in terms of x and y.

It follows readily from the definition of inverse that if G transforms the region D_0 in the uv-pane to the region D in the xy-plane, then F transforms D into D_0. This will often helps us determine the domain of integration after changing variables.

Example 2. Consider Example 1 from yesterday's lecture. We have $G(u, v) = (uv^{-1}, uv)$, and we showed that G transforms the rectangle $R = [1, 2] \times [1, 2]$ in the uv-plane to the region D in the xy-plane bounded by two lines and two hyperboles. Given D how does one find R? If we can find $F(x, y)$, then we apply F to D to get R. Since $x = uv^{-1}, u = vx$. Substituting this into the equation $y = uv$, we get $y = (vx)v = v^2x$.

Thus, $v = \sqrt{\frac{y}{x}}$. Note that we take the positive square root here (by choice). Since $u = vx$, $u = \sqrt{\frac{y}{x}}x = \sqrt{xy}$. Thus, $F(x,y) = (\sqrt{xy}, \sqrt{\frac{y}{x}})$. Now let's see how F transforms D into R.

When $xy = 1$, $x = \frac{1}{y}$, so that $(u,v) = F(x,y) = F(x, \frac{1}{y}) = (\sqrt{1}, \sqrt{\frac{y}{\frac{1}{y}}}) = (1, y)$. Keeping in mind the first coordinate is u and the second is v we see that F transforms the hyperbole $xy = 1$ into the line $u = 1$. The portion of $xy = 1$ along the boundary of D occurs for $1 \leq y \leq 2$, so we get the portion of $u = 1$, for $1 \leq v \leq 2$, in other words, the left edge of R. Similarly, if we consider $F(x,y)$ along $xy = 4$, so that $x = \frac{y}{4}$, we have $(u,v) = F(\sqrt{4}, \sqrt{\frac{y}{\frac{4}{y}}}) = (2, \frac{1}{2}y)$, which is the line $u = 2$ in the uv plane. Since along the hyperbole $xy = 4$, we have $2 \leq y \leq 4$, it follows that $1 \leq v \leq 2$, so we obtain the right edge of R. When $y = 4x$, $(u,v) = F(x, 4x) = (\sqrt{4x^2}, \sqrt{\frac{4x}{x}}) = (2x, 2)$, which in the uv plane is the line $v = 2$. Since along the boundary of D, the line $y = 4x$ varies as $.5 \leq x \leq 1$, $1 \leq u \leq 2$, giving the top segment of R. A similar analysis shows that the portion of the boundary of D determined by $y = x$ becomes the bottom edge of R. Thus, F transforms D into R. So that if we were given D, and did not know R, this method finds D using inverse transformation F.

Example 3. A similar analysis is even easier for Example 3 from yesterday's lecture, though the difference is that $G(u,v)$ was not originally given to us. We expressed u, v in terms of x, y which we now know is the same as giving F, as a transformation from the xy-plane to the uv-plane, whose inverse will be the G we want. From last lecture we have $(u,v) = F(x,y) = (x+y, x-y)$, which enabled us to see that $G(u,v) = (\frac{u+v}{2}, \frac{u-v}{2})$. Moreover, since the edges of D are given by the equations $x + y = 1, x + y = -1, x - y = 1, x - y = -1$, in terms of u and v we have $u = 1, u = -1, v = 1, v = -1$, which tells us that D_0 should be the rectangle $[1,1] \times [-1,1]$. In more detail, we can use F. So for example, along the line edge of D given by $x + y = 1$, $y = 1 - x$, so that $(u,v) = F(x, 1-x) = (x + (1-x), x - (1-x)) = (1, -1 + 2x)$, which in the uv plane is the line $u = 1$. The portion of the line $x + y = 1$ along D. occur for $0 \leq x \leq 1$, so that on D_0, the line segment runs from $-1 + 2 \cdot 0 = -1$ to $-1 + 2 \cdot 1 = 1$, which gives the left edge of D_0. Similarly one can use F to see that each of the edges of D go to the edges of D_0. Thus, if we did know know D_0, applying F to D tellls us what D_0 should be.

Example 4. Suppose that $T(u,v) = (au + bv, cu + dv)$ is a linear transformation with non-zero Jacobian $ad - bc$. Set $\delta = ad - bc$. Then

$$T^{-1}(x,y) = \frac{1}{\delta} \cdot (dx - by, -cx + ay).$$

T^{-1} can be gotten by solving for u and v in terms of x and y in the system of equations

$$x = au + bv$$
$$y = cx + dv$$

using Cramer's rule. Note that

$$\text{Jac}(T^{-1}) = \det \begin{pmatrix} \frac{d}{\delta} & -\frac{b}{\delta} \\ -\frac{c}{\delta} & \frac{a}{\delta} \end{pmatrix} = \frac{ad - bc}{\delta^2} = \frac{1}{\delta}.$$

In other words, $\text{Jac}(T^{-1}) = \frac{1}{\text{Jac}(T)}$. Thus phenomenon holds in general.

Theorem. Let G be a one-to-one transformation (i.e, G has continuous first order partial derivatives) from $\mathbb{R}^2$ to $\mathbb{R}^2$, with inverse F. Then $\text{Jac}(F) = \frac{1}{\text{Jac}(G)}$.

We finished the lecture with a discussion explaining how the Jacobian comes into the change of variables formula for double integrals.

Why do we use $dA = |\frac{\partial(x,y)}{\partial(u,v)}| dudv$? To see this, start with the transformation $G(u,v) = (x(u,v), y(u,v))$. For this, we need two facts:

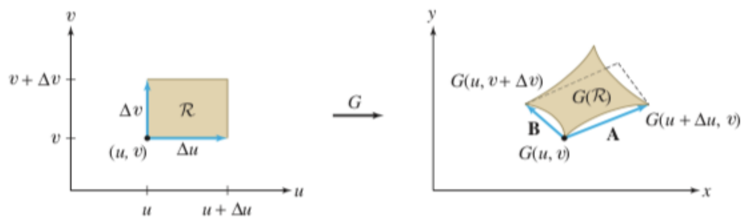
(i) If $A = a\vec{i} + b\vec{j}$ and $B = c\vec{i} + d\vec{j}$, then the area of the parallelogram spanned by A and B is the absolute value of $\det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = |ad - bc|$.

(ii) For a function $f(u,v)$ whose partial derivatives exist:

$$f(u+\Delta u, v) - f(u,v) \approx \Delta u \frac{\partial f}{\partial u} \quad \text{and} \quad f(u, v+\Delta v) - f(u,v) \approx \Delta v \frac{\partial f}{\partial v},$$

when Δu and Δv are small. This follows, since for example, $\frac{\partial f}{\partial u} \approx \frac{f(u+\Delta u, v)-f(u,v)}{\Delta u}$.

Now, G transforms the rectangle with area $\Delta u \Delta v$ to the curvilinear rectangle shown below:



In the Riemann sums of the double integral in x and y over the region $G(\mathcal{R})$, we may use the parallelogram P spanned by spanned by the vectors $\mathbf{A}$ and $\mathbf{B}$ as small portions of area dA. Note that

$$\begin{aligned} \mathbf{A} &= (x(u+\Delta u, v) - x(u,v))\ \vec{i} + (y(u+\Delta u, v) - y(u,v))\ \vec{j} \\ &\approx \Delta u \frac{\partial x}{\partial u}\ \vec{i} + \Delta v \frac{\partial x}{\partial v}\ \vec{j}. \end{aligned}$$

Similarly: $\mathbf{B} \approx \Delta v \frac{\partial x}{\partial v}\ \vec{i} + \Delta v \frac{\partial y}{\partial v}\ \vec{j}$. Therefore:

$$\begin{aligned} dA &\approx \text{area}(\mathcal{R}) \\ &\approx \text{area}(P) \\ &\approx \left| \det \begin{pmatrix} \Delta u \frac{\partial x}{\partial u} & \Delta v \frac{\partial x}{\partial v} \\ \Delta u \frac{\partial y}{\partial u} & \Delta v \frac{\partial y}{\partial v} \end{pmatrix} \right| \\ &= \left| \Delta u \Delta v \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \Delta u \Delta v \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right| \\ &= \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v. \end{aligned}$$

Thus, the Riemann sum: $\sum_i \sum_j f(x_i, y_j)\ dA$ in xy-coordinates is approximately the Riemann sum

$$\sum_i \sum_j f(x(u_j, v_j), y(u_i, v_j)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v,$$

in uv-coordinates. Passing to the limit as the units of area tend to zero gives the change of variables formula.

Wednesday, March 24. Today we worked in breakout rooms on the practice problems from the Exam 2 review sheet.

Thursday, March 25. Today we discussed solutions to some of the practice problems on the Exam 2 review sheet. Full solutions were posted on the course webpage.

Friday, March 26. Exam 2.

Monday, March 29. We began class by outlining solutions to Exam 2. the full set of solutions are posted on our course web page. We then began a discussion of triple integrals.

A triple integral is an integral of the form $\int\int\int_B f(x,y,z)\ dV$, where B is a solid region contained in $\mathbb{R}^3$. The underlying idea for the definition of $\int\int\int_B f(x,y,z)\ dV$ is the same as we discussed for double integrals:

First, partition the domain of integration B into small subregions - in this case solids - of a similar type.

Second, select a point from each small subregion and evaluate the function $f(x,y,z)$ at that point.

Third, multiply the value obtained in the second step by the size of the subregion the point was chosen from. In this case we are multiplying the function value by a small unit of volume.

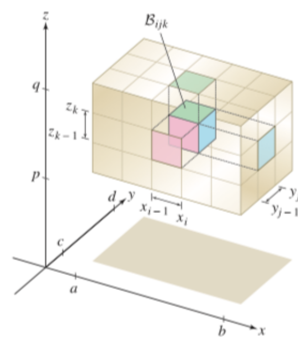
Fourth, add up the values from the previous step, thereby getting a Riemann sum.

Fifth, take a limit of the Riemann sums as the maximum volumes of the subregions in the partition go to zero.

Sixth, if the limit exists, we denote it $\int\int\int_B f(x,y,z)\ dV$.

As in previous discussions concerning double integrals, $\int\int\int_B f(x,y,z)\ dV$ is a quantity that depends upon $f(x,y,z)$ and the geometry of B and does not depend upon the coordinate system used to describe B or used to calculate $\int\int\int_B f(x,y,z)\ dV$.

As expected, we have various versions Fubini's Theorem for triple integrals. If B is a rectangular box, and we use rectangular coordinates, our Riemann sums look something like this, which justifies Fubini's Theorem:



$$S_{N,M,L} = \sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{k=1}^{L} f(P_{ijk})\,\Delta V_{ijk}$$

Fubini's Theorem for rectangular boxes. Suppose $B = [a,b]\times[c,d]\times[p,q]$ is a rectangular box in $\mathbb{R}^3$ and $f(x,y,z)$ is continuous on B. Then:

$$\int\int\int_B f(x,y,z)\ dV = \int_a^b\int_c^d\int_p^q f(x,y,z)\ dzdydx.$$

Moreover, $\int\int\int_B f(x,y,z)\ dV$ can be calculated in any one of the five remaining ways to permute the order of integration. For example:

$$\begin{aligned}\int\int\int_B f(x,y,z)\ dV &= \int_c^d\int_p^q\int_a^b f(x,y,z)\ dxdzdy\\ &= \int_p^q\int_a^b\int_c^d f(x,y,z)\ dydxdz.\end{aligned}$$

Example 1. Calculate $\int\int\int_B x^2 + 2yz \; dV$, where $B = [0,1] \times [-1,0] \times [1,2]$.

Solution: Applying Fubini's Theorem,

$$
\begin{aligned}
\int\int\int_B x^2 + 2yz \; dV &= \int_0^1 \int_{-1}^0 \int_1^2 (x^2 + 2yz) \; dzdydx \\
&= \int_0^1 \int_{-1}^0 (x^2 z + yz^2)_{z=1}^{z=2} \; dydx \\
&= \int_0^1 \int_{-1}^0 (2x^2 + 4y) - (x^2 + y) \; dydz \\
&= \int_0^1 \int_{-1}^0 x^2 + 3y \; dydx \\
&= \int_0^1 (x^2 y + \frac{3}{2} y^2)_{y=-1}^{y=0} \; dx \\
&= \int_0^1 (x^2 - \frac{3}{2}) \; dx \\
&= (\frac{x^3}{3} - \frac{3x}{2})_{x=0}^{x=1} \\
&= \frac{1}{3} - \frac{3}{2} = -\frac{7}{6}.
\end{aligned}
$$

Integrating in a different order we get:

$$
\begin{aligned}
\int\int\int_B x^2 + 2yz \; dV &= \int_{-1}^0 \int_1^2 \int_0^1 (x^2 + 2yz) \; dxdzdy \\
&= \int_{-1}^0 \int_1^2 (\frac{x^3}{3} + 2xyz)_{x=0}^{x=1} \; dzdy \\
&= \int_{-1}^0 \int_1^2 (\frac{1}{3} + 2yz) \; dzdy \\
&= \int_{-1}^0 (\frac{z}{3} + yz^2)_{z=1}^{z=2} \; dy \\
&= \int_{-1}^0 \frac{1}{3} + 3y \; dy \\
&= (\frac{y}{3} + \frac{3y^2}{2})_{y=-1}^{y=0} \\
&= 0 - (-\frac{1}{3} + \frac{3}{2}) = -\frac{7}{6}.
\end{aligned}
$$

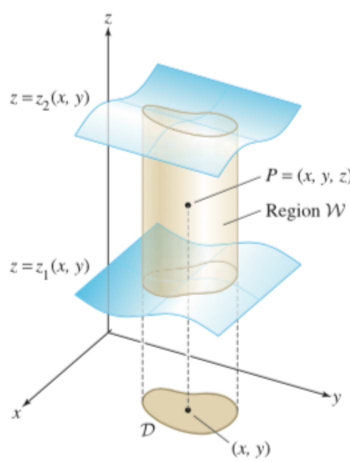
Fundamental Fact: $\int\int\int_B \; dV = \text{vol}(B)$. This follows, since in the Riemann sums approximating $\int\int\int_B \; dV$, we are just adding volumes of small solids covering B, so that each Reimann sum provides a better approximation to the volume of B and by passing to the limit, we obtain the volume of B.

Example 2. Let B denote the box in Example 1. Then:

$$\begin{aligned}\int\int\int_B dV &= \int_0^1\int_{-1}^0\int_1^2 dzdydx\\ &= \int_0^1\int_{-1}^0 (2-1)\ dydx\\ &= \int_0^1\int_{-1}^0\ dydx\\ &= \int_0^1 (0-(-1))\ dx\\ &= \int_0^1\ dx\\ &= 1.\\ &= \text{vol}(B)\end{aligned}$$

as expected.

What about more general domains of integration? Suppose we have a region $\mathcal{W} \subseteq \mathbb{R}^3$ defined as all points (x, y, z) in $\mathbb{R}^3$ such that $(x, y) \in D$ and $z_1(x, y) \leq z \leq z_2(x, y)$, as pictured below.



We have a corresponding version of Fubini's Theorem:

$$\int\int\int_{\mathcal{W}} f(x,y,z)\ dV = \int\int_D \{ \int_{z(_1x,y)}^{z_2(x,y)} f(x,y,z)\ dz \}\ dA.$$

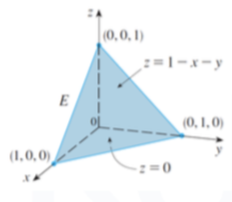
Note that if we calculate the integral in the brackets by integrating with respect to z, and then evaluating z at $z_2(x, y)$ and $z_1(x, y)$ and subtracting, we then have a double integral of a function in x and y only over the domain $D \subseteq \mathbb{R}^2$. In this version of Fubini's Theorem, we can reduce a triple to a double integral.

Tuesday, March 30. We began class with Quiz 6, then continued our discussion of triple integrals, starting by recalling the fact that that if B is the solid rectangular box $[a, b] \times [c, d] \times [p, q]$, $\int\int\int_B f(x, y, z)\ dV$ can be calculated in six different ways as an iterated integral, one of which is $\int_p^q \int_c^d \int_a^b f(x, y, z)\ dz\ dy\ dx$, the other five ways obtained by the various permutations of dx, dy, dz. We then recalled the formula from the last lecture, granted by a second form of Fubini's Theorem for triple integrals that if B is given as all points (x, y, z) in $\mathbb{R}^3$ such that $(x, y) \in D$ and $z_1(x, y) \leq z \leq z_2(x, y)$, then,

$$\int\int\int_{\mathcal{B}} f(x,y,z)\ dV = \int\int_D \{ \int_{z(_1x,y)}^{z_2(x,y)} f(x,y,z)\ dz \}\ dA.$$

We refer to B as defined above as a *z-simple region* in $\mathbb{R}^3$. We then worked the following example.

Example 1. Find $\int\int\int_B e^{x+y+z}\ dV$ for B the tetrahedron below:



Solution. Regarding B as a z-simple region, we have that $0 \leq z \leq 1-x-y$ and D is the triangle $0 \leq x \leq 1$, $0 \leq y \leq 1-x$. Thus:

$$\begin{aligned}
\int\int\int_B e^{x+y+z}\ dV &= \int\int_D \{ \int_0^{1-x-y} e^{x+y+z}\ dz \}\ dA. \\
&= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} e^{x+y+z}\ dzdydx \\
&= \int_0^1 \int_0^{1-x} e^{x+y+z}\Big|_{z=0}^{z=1-x-y}\ dydx \\
&= \int_0^1 \int_0^{1-x} (e^1 - e^{x+y})\ dydx \\
&= \int_0^1 (ey - e^{x+y})_{y=0}^{y=1-x}\ dx \\
&= \int_0^1 (e(1-x) - e) - (0 - e^x)\ dx \\
&= \int_0^1 (-ex + e^x)\ dx \\
&= (-\frac{e}{2}x^2 + e^x)_0^1 \\
&= (-\frac{e}{2} + e) - (0+1) \\
&= \frac{e}{2} - 1.
\end{aligned}$$

We then noted the following. Suppose B is a z-simple region defined over the domain $D \subseteq \mathbb{R}^2$. If D is a region of Type 1, say $c(x) \leq y \leq d(x)$, $a \leq x \leq b$, then

$$\int\int\int_{\mathcal{B}} f(x,y,z)\ dV = \int_a^b \int_{c(x)}^{d(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\ dz\ dy\ dx,$$

while if D is a region of Type 3 described as: $a(y) \leq x \leq b(y)$ and $c \leq y \leq d$, then

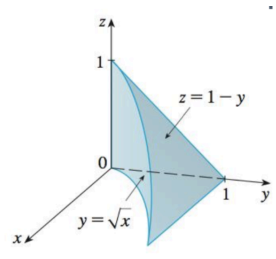
$$\int\int\int_{\mathcal{B}} f(x,y,z)\ dV = \int_c^d \int_{a(y)}^{b(y)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\ dz\ dx\ dy.$$

It may also be that D is a region more easily described in terms of polar coordinates, in which case, we calculate $\int\int\int_B f(x,y,z)\ dV$ by first integrating with respect to z, and then use polar coordinates. If $F(x,y,z)$ is an antiderivative of $f(x,y,z)$ with respect to z, and D is given by $r_1(\theta) \leq r \leq r_2(\theta)$ and $\theta_1 \leq \theta \leq \theta_2$, then $\int\int\int_B f(x,y,z)\ dV =$

$$\int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \{F(r\cos(\theta), r\sin(\theta), z_2(r\cos(\theta), r\sin(\theta))) - F(r\cos(\theta), r\sin(\theta), z_1(r\cos(\theta), r\sin(\theta)))\}\ r\ dr\ d\theta.$$

Wednesday, March 31. We continued with our discussion of triple integrals, beginning with the following example.

Example 1. Calculate $\int\int\int_B x \; dV$ for B

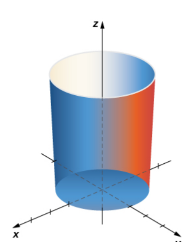


Solution

$$\begin{aligned}
\int\int\int_B x \; dV &= \int\int_D \{ \int_0^{1-y} x \; dz \} \; dA \\
&= \int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} x \; dzdydx \\
&= \int_0^1 \int_{\sqrt{x}}^1 xz\Big|_{z=0}^{z=1-y} \; dydx \\
&= \int_0^1 \int_{\sqrt{x}}^1 x(1-y) \; dydx \\
&= \int_0^1 x(y - \frac{y^2}{2})_{y=\sqrt{x}}^{y=1} \; dx \\
&= \int_0^1 x(\frac{1}{2} - \sqrt{x} + \frac{x}{2}) \; dx \\
&= (\frac{x^2}{4} - \frac{2}{5}x^{\frac{5}{2}} + \frac{x^3}{6})_0^1 \\
&= \frac{1}{4} - \frac{2}{5} + \frac{1}{6} \\
&= \frac{1}{60}
\end{aligned}$$

We then recalled that for a solid region $B \subseteq \mathbb{R}^3$, volume$(B) = \int\int\int_B \; dV$. We demonstrated this with the following example.

Example 2. Use a triple integral to find the volume of a cylinder B of radius R and height h.



Here B is bounded above by the plane $z = h$ and below by the plane $z = 0$. The domain D in the xy-plane is: $0 \le x^2 + y^2 \le R^2$. Thus:

$$\begin{aligned}
\text{vol}(B) &= \int\int\int_B dV \\
&= \int\int_D \int_0^h dx\ dA \\
&= \int_0^{2\pi} \int_0^R \int_0^h dz\ r dr d\theta \\
&= \int_0^{2\pi} \int_0^R hr\ dr d\theta \\
&= \int_0^{2\pi} h\frac{R^2}{2}\ d\theta \\
&= 2\pi\frac{R^2}{2}h \\
&= \pi R^2 h,
\end{aligned}$$

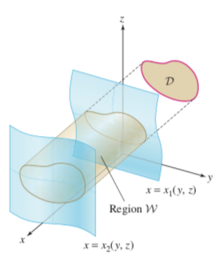
as expected. We also noted that if B is a z-simple region defined by $z_1(x, y) \le z \le z_2(x, y)$ and $(x, y) \in D$, then

$$\begin{aligned}
\text{vol}(B) &= \int\int\int_B dV \\
&= \int\int_D \left\{ \int_{z(_1x,y)}^{z_2(x,y)} dz \right\} dA \\
&= \int\int_D \left\{ z\Big|_{z=z_1(x,y)}^{z=z_2(x,y)} \right\} dA \\
&= \int\int_D z_2(x, y) - z_1(x, y)\ dA,
\end{aligned}$$

which is our previous formula for the volume of the region in $\mathbb{R}^3$ defined over D and bounded above and below by the surfaces $z = z_2(x, y)$ and $z = z_1(x, y)$.

We then turned to the consideration of triple integrals over regions that are not z-simple, but those that live above or below the yz-plane or xz-plane.

We have two other types of simple solids. An x-simple solid has the form $x_1(y, z) \le x \le x_2(y, z)$, $(y, z) \in D$, with D in the yz-plane.



By Fubini's Theorem:

$$\int\int\int_{\mathcal{W}} f\ dV = \int\int_D \left\{ \int_{x_1(y,z)}^{x_2(y,z)} f\ dx \right\} dA.$$

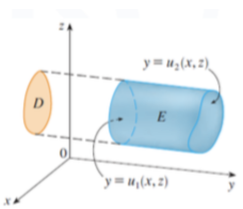
If D is described as: $a(z) \le y \le b(z)$ and $c \le z \le d$, then

$$\int\int\int_{\mathcal{W}} f\ dV = \int_c^d \int_{a(z)}^{b(z)} \int_{x_1(y,z)}^{x_2(y,z)} f\ dxdydz.$$

While if D is described as: $c(y) \le z \le d(y)$ and $a \le y \le b$, then

$$\int\int\int_{\mathcal{W}} f\ dV = \int_a^b \int_{c(y)}^{d(y)} \int_{x_1(y,z)}^{x_2(y,z)} f\ dxddzdy.$$

A y-simple solid E has the form $u_1(x,z) \le y \le u_2(x,z)$, $(y,z) \in D$, with D in the xz-plane.



By Fubini's Theorem:

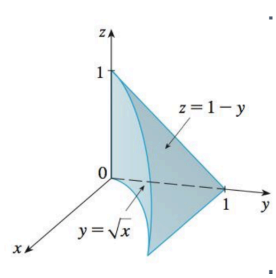
$$\int\int\int_E f\ dV = \int\int_D \{ \int_{u_1(x,z)}^{u_2(x,z)} f\ dx \}\ dA.$$

which can take the form

$$\int_c^d \int_{a(x)}^{b(x)} \int_{u_1(x,z)}^{u_2(x,z)} f\ dydzdx \quad \text{or} \quad \int_a^b \int_{c(z)}^{d(z)} \int_{u_1(x,z)}^{u_2(x,z)} f\ dydxdz$$

We ended class by revisiting the first example, viewing it as a y-simple region.

Example 3. Calculate $\int\int\int_B x\ dV$ for B below, viewed as a y-simple surface.

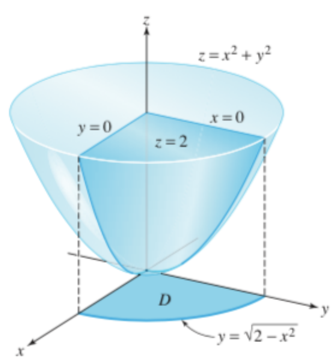


Solution. Rewriting the plane $z = 1 - y$ as $y = 1 - z$, we see that if we move from left to right along the y-axis, the region B is bounded to the left by the surface $y = \sqrt{x}$ and to the right by the surface $y = 1 - z$, which then become the lower and upper limits of the inner most iterated integral. Where these surfaces intersect determines the region D in the xz-plane that we integrate over. Since $y = \sqrt{x}$ and $z = 1 - y$, we have $z = 1 - \sqrt{x}$. It follows that in the xz-plane, D is determined by $0 \leq z \leq 1 - \sqrt{x}$ and $0 \leq x \leq 1$. Thus,

$$\begin{aligned}
\int\int\int_B x\; dV &= \int\int_D \left\{ \int_{\sqrt{x}}^{1-z} x\; dy \right\} dA \\
&= \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} x\; dydzdx \\
&= \int_0^1 \int_0^{1-\sqrt{x}} xy\Big|_{y=\sqrt{x}}^{y=1-z} dzdx \\
&= \int_0^1 \int_0^{1-\sqrt{x}} x(1 - z - \sqrt{x})\; dzdx \\
&= \int_0^1 (xz - \frac{xz^2}{2} - \sqrt{x}z)_{z=0}^{z=(1-\sqrt{x})}\; dx \\
&= \int_0^1 x(1-\sqrt{x}) - \frac{x}{2}(1 - 2\sqrt{x} + x) - \sqrt{x}(1-\sqrt{x})\; dx \\
&= \cdots \\
&= \frac{1}{60}
\end{aligned}$$

Thursday, April 1. We began by reviewing the form $\int\int\int_B f(x, y, z)\; dV$ takes when viewing B as a x-simple, y-simple, or z-simple region in $\mathbb{R}^3$. We then worked several examples, primarily focusing on setting up the iterated integrals.

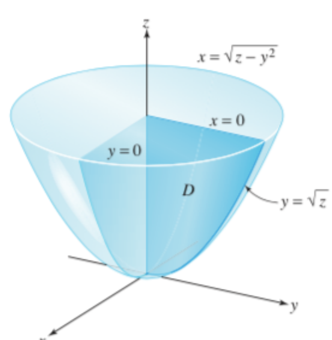
Example 1a. Calculate $\int\int\int_B x\; dV$ for B viewing B as a z-simple region.



Solution. Viewing B as a z-simple region, we have

$$\begin{aligned}
\int\int\int_B x\ dV &= \int\int_D \int_{x^2+y^2}^{2} x\ dz\ dA \\
&= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{x^2+y^2}^{2} x\ dzdydx \\
&= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} xz\Big|_{z=x^2+y^2}^{z=2} dydx \\
&= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} x(2-x^2-y^2)dydx \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} r\cos(\theta)(2-r^2)r\ drd\theta \\
&= \int_0^{\frac{\pi}{2}} \cos(\theta) \int_0^{\sqrt{2}} 2r^2 - r^4\ drd\theta \\
&= \int_0^{\frac{\pi}{2}} \cos(\theta)\cdot(\frac{2r^3}{3} - \frac{r^5}{5})_{r=0}^{r=\sqrt{2}}\ d\theta \\
&= \int_0^{\frac{\pi}{2}} \cos(\theta)\cdot(\frac{4\sqrt{2}}{3} - \frac{4\sqrt{2}}{5})\ d\theta \\
&= \frac{8\sqrt{2}}{15}\int_0^{\frac{\pi}{2}} \cos(\theta)\ d\theta \\
&= \frac{8\sqrt{2}}{15}.
\end{aligned}$$

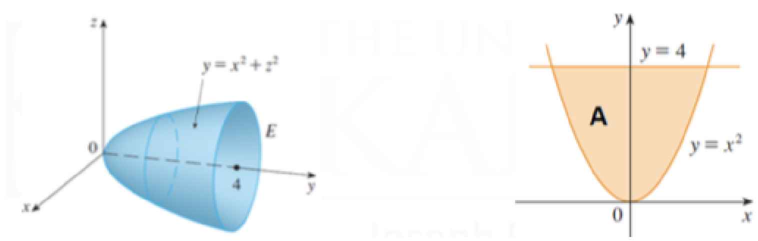
Example 1b. The same integral integrating x first.



Note that $0 \le x \le \sqrt{z-y^2}$ and the domain D in the yz-plane is bounded below by the parabola $z = y^2$ and above by $z = 2$. Thus

$$\begin{aligned}
\int\int\int_B x\ dV &= \int_0^{\sqrt{2}}\int_{y^2}^{2}\int_0^{\sqrt{z-y^2}} x\ dxdzdy \\
&= \frac{1}{2}\int_0^{\sqrt{2}}\int_{y^2}^{2} z - y^2\ dzdy \\
&= \frac{1}{2}\int_0^{\sqrt{2}} (\frac{z^2}{2} - y^2 z)_{z=y^2}^{z=2}\ dy \\
&= \frac{1}{2}\int_0^{\sqrt{2}} (2-2y^2) - (\frac{y^4}{2} - y^4)\ dy \\
&= \frac{1}{2}\int_0^{\sqrt{2}} 2 - 2y^2 + \frac{y^4}{2}\ dy \\
&= \frac{1}{2}(2y - \frac{2y^3}{3} + \frac{y^5}{10})_0^{\sqrt{2}} \\
&= \frac{1}{2}\{2\sqrt{2} - \frac{4\sqrt{2}}{3} + \frac{4\sqrt{2}}{10}\} \\
&= \frac{8\sqrt{2}}{15}.
\end{aligned}$$

Example 2. This example shows how changing the order of integration can simplify the integration. Consider $\int\int\int_E \sqrt{x^2+z^2}\ dV$ for the solid E pictured on the left



Regarding E as a z-simple region, with domain of integration A, we have

$$\begin{aligned}
\int\int\int_E \sqrt{x^2+z^2}\ dV &= \int\int_A\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\ dz\ dA \\
&= \int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\ dzdyx \\
&= \int_0^4\int_{-\sqrt{y}}^{\sqrt{y}}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\ dzdxdy,
\end{aligned}$$

which is difficult to integrate

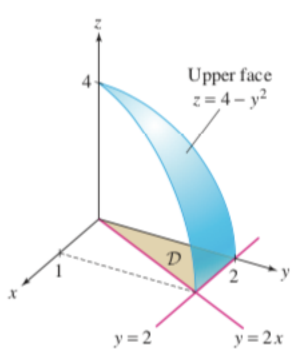
On the other hand, regarding E as a y-simple region with domain of integration , we tegrate with respect to y first, to obtain

$$\int\int\int_E \sqrt{x^2+z^2}\ dV = \int\int_B \int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\ dy\ dB$$
$$= \int\int_B 4\sqrt{x^2+z^2} - (x^2+z^2)\sqrt{x^2+z^2}\ dB$$
$$= \int_0^{2\pi}\int_0^2 (4r - r^3)r\ drd\theta$$
$$= 2\pi\int_0^2 4r^2 - r^4\ dr$$
$$= 2\pi(\frac{32}{3} - \frac{32}{5})$$
$$= \frac{128\pi}{15}.$$

We then had the class set up the following triple integral.

Example 3.a Set up $\int\int\int_B xyz\ dV$ where B is the z-simple region given by

$$z = 4y^2, y = 2x, z = 0, x = 0.$$

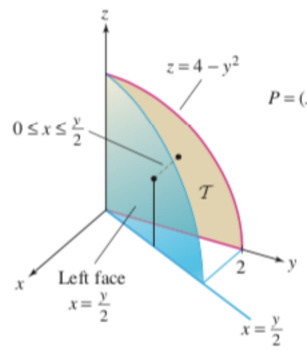


(A) Projection to xy-plane

Solution:

$$\int\int\int_B xyz\ dV = \int\int_D \{\int_0^{4-y^2} xyz\ dz\}\ dA = \int_0^1\int_{2x}^2\int_0^{4-y2} xyz\ dzdydx.$$

Example 3b. Set up the triple integral for the previous example in the order x, z, y, viewing B as an x-simple surface.



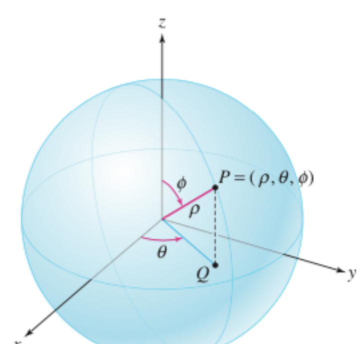
(B) Projection to yz-plane

Solution:

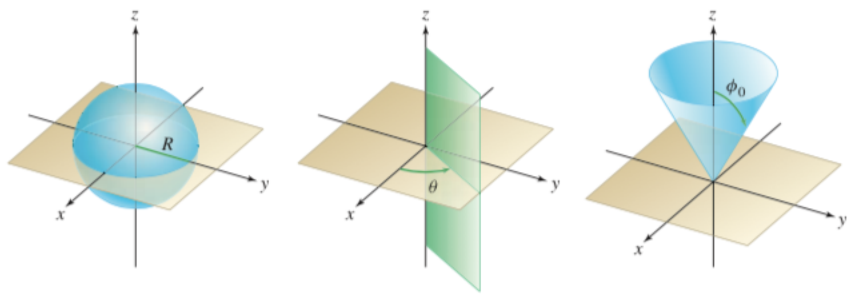
$$\int\int\int_B xyz\ dV = \int\int_T \{\int_0^{\frac{y}{2}} xyz\ dx\}\ dA = \int_0^2 \int_0^{4-y^2} \int_0^{\frac{y}{2}} xyz\ dxdzdy$$

We ended class by considering the integral $\int\int\int_B \sqrt{x^2+y^2+z^2}\ dV$, where B is the solid sphere of radius one centered at the origin. We noted that this requires a three dimensional version of polar coordinates, and suggested that a substitution using spherical coordinates should be the analogous procedure. Examples of this type will be discussed in the next lecture.

Friday, April 2. Today we reviewed spherical coordinates as a technique for calculating integrals of the form $\int\int\int_B \sqrt{x^2+y^2+z^2}$, where $B \subseteq \mathbb{R}^3$ is the solid sphere of radius one centered at the origin. We began by observing that every point in $\mathbb{R}^3$ lies on a sphere of radius ρ centered at (0,0,0) and thus can be expressed in terms of spherical coordinates, ρ, ϕ, θ. Here is typical point P using spherical coordinates

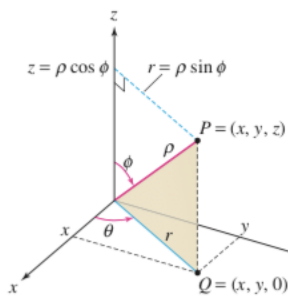


Note: $P = (\rho, \phi, \theta)$, with $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$. If we set each of the spherical coordinate equal to a constant, we get:



$\rho = R$ Sphere of radius R | $\theta = \theta_0$ Vertical half-plane | $\phi = \phi_0$ Right-circular cone

We then showed the relation between the spherical coordinates and the rectangular coordinates of P.



Note:

$$x = r\cos(\theta) = \rho\sin(\phi)\cos(\theta), y = r\sin(\theta) = \rho\sin(\phi)\sin(\theta), z = \rho\cos(\phi).$$

It follows from these equations that the expression $x^2 + y^2 + z^2$, in spherical coordinates, becomes ρ^2. We will use this fact often. Writing spherical coordinates in terms of rectangular coordinates.

$\rho = \sqrt{x^2 + y^2 + z^2}$
$\tan(\theta) = \frac{y}{x}$, so $\theta = \tan^{-1}(\frac{y}{x})$.
$\cos(\phi) = \frac{z}{\rho}$, so $\phi = \cos^{-1}(\frac{z}{\rho})$.

Example 1. Find the rectangular coordinates of the point $P = (\rho, \phi, \theta) = (3, \frac{\pi}{3}, \frac{\pi}{4})$.

Solution

$$x = 3\sin(\frac{\pi}{3})\cos(\frac{\pi}{4}) = 3 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{3\sqrt{6}}{4}$$
$$y = 3\sin(\frac{\pi}{3})\sin(\frac{\pi}{4}) = 3 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{3\sqrt{6}}{4}$$
$$z = 3\cos(\frac{\pi}{3}) = 3 \cdot \frac{1}{2} = \frac{3}{2}.$$

Example 2. Find the spherical coordinates of the point $P = (x, y, z) = (-1, 1, \sqrt{6})$.

Solution. $\rho = \sqrt{(-1)^2 + 1^2 + (\sqrt{6})^2} = \sqrt{8} = 2\sqrt{2}$. From $z = \rho\cos(\phi)$ we have $\sqrt{6} = 2\sqrt{2}\cos(\phi)$.Thus, $\cos(\phi) = \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{3}}{2}$. $\phi = \frac{\pi}{6}$. $\theta = \tan^{-1}(\frac{-1}{1}) = \tan^{-1}(-1) = \frac{3\pi}{4}$, since $(x, y) = (-1, 1)$. Thus, in spherical coordinates, $P = (2\sqrt{2}, \frac{\pi}{6}, \frac{3\pi}{4})$.

When then saw that to convert a triple integral $\int\int\int_B f(x,y,z)\ dV$ into an iterated integral involving spherical coordinates, there are two step involved. First, one describes B in terms of spherical coordinates, as follows: $\rho_1(x,y) \le \rho \le \rho_2(x,y)$, $\phi_1 \le \pi \le \phi_2$, $\theta_1 \le \theta \le \theta_2$. We then set

$$x = \rho\sin(\phi)\cos(\theta), y = \rho\sin(\phi)\sin(\theta), z = \rho\cos(\phi) \quad \text{and} \quad dV = \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta,$$

with the understanding that we will explain the equality $dV = \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$ in our next lecture. Thus, by Fubini's Theorem, we have

$$\int\int\int_B f(x,y,z)\ dV = \int_{\theta_1}^{\theta_2}\int_{\rho_1}^{\phi_2}\int_{\rho_1(\phi,\theta)}^{\rho_2(\phi,\theta)} f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\ \rho^2\sin(\phi)\ d\rho d\phi d\theta.$$

We then worked the example mentioned at the outset.

Example 3. Calculate $\int\int\int_B \sqrt{x^2+y^2+z^2}$, where $B \subseteq \mathbb{R}^3$ is the solid sphere of radius one centered at the origin.

Solution. We first noted that in terms of spherical coordinates, B can be described by

$$0 \le \rho \le 1, \quad 0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi.$$

Thus,

$$\begin{aligned}
\int\int\int_B \sqrt{x^2+y^2+z^2}\ dV &= \int\int\int_B \sqrt{\rho^2\sin^2(\phi)\cos^2(\theta)+\rho^2\sin^2(\phi)\sin^2(\theta)+\rho^2\cos^2(\phi)}\ \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^3\sin(\phi)\ d\rho\ d\phi\ d\theta \\
&= \frac{1}{4}\int_0^{2\pi}\int_0^{\pi}\sin(\phi)\ d\phi\ d\theta \\
&= \frac{2\pi}{4}\int_0^{\pi}\sin(\phi)\ d\phi \\
&= \frac{2\pi}{4}\cdot 2 \\
&= \pi.
\end{aligned}$$

We ended class with the following interesting interpretation of the integral above. If $B \subseteq \mathbb{R}^3$ is a solid with finite volume, and $f(x,y,z)$ is continuous on B, then, as expected,

$$\text{The average value of } f(x,y,z) \text{ over } B = \frac{1}{\text{vol}(B)}\int\int\int_B f(x,y,z)\ dV.$$

If we take B to be the solid sphere of radius one, centered at the origin, then the distance of any point (x,y,z) in B to the origin is $\sqrt{x^2+y^2+z^2}$. Thus, the average distance of points in B to the origin is

$$\frac{1}{\text{vol}(B)}\int\int\int_B \sqrt{x^2+y^2+z^2}\ dV = \frac{3}{4\pi}\cdot\pi = \frac{3}{4}.$$

Monday, April 5. We continued our discussion of calculating triple integrals using spherical and cylindrical coordinates, recalling that in spherical coordinates,

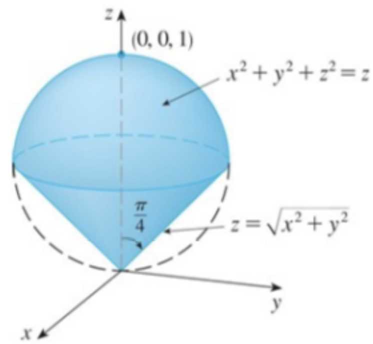
$$x = \rho\sin(\phi)\cos(\theta), y = \rho\sin(\phi)\sin(\theta), z = \rho\cos(\phi) \quad \text{and} \quad dV = \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta,$$

We began by using spherical coordinates to calculate the volume of a sphere of radius R.

Example 1. If B denotes the sphere of radius R centered at the origin, then in spherical coordinates, B is given by $0 \le \rho \le R$, $0 \le \phi \le \pi, 0 \le \theta \le 2\pi$. Thus,

$$\begin{aligned}
\text{vol}(B) &= \int\int\int_B dV \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin(\phi) d\rho d\phi d\theta \\
&= \frac{2\pi R^3}{3}\int_0^{\pi} \sin(\phi)\ d\phi \\
&= \frac{4\pi R^3}{3}.
\end{aligned}$$

Exampel 2. Calculate $\int\int\int_B \sqrt{x^2+y^2}\ dV$ for the solid B bounded by the cone $z = \sqrt{x^2+y^2}$ and the sphere $z = z^2 + y^2 + z^2$.



For the sphere:

$$\rho\cos(\phi) = z = x^2 + y^2 + x^2 = \rho^2,$$

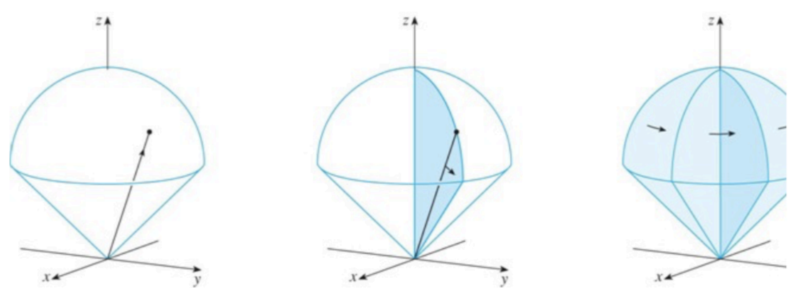
So $\rho = \cos(\phi)$ is the equation of the sphere. Thus, for B, we have $0 \le \rho \le \cos(\phi)$.

The cone:

$$\begin{aligned}
\rho\cos(\phi) &= \sqrt{(\rho\sin(\phi)\cos(\theta))^2 + (\rho\sin(\phi)\sin(\theta))^2} \\
&= \rho\sin(\phi),
\end{aligned}$$

so $\cos(\phi) = \sin(\phi)$, which gives $\phi = \frac{\pi}{4}$, for the equation of the cone.

Where the cone and sphere meet: When $\phi = \frac{\pi}{4}$ on the sphere. Therefore, for B, we have $0 \le \phi \le \frac{pi}{4}$. θ varies from 0 to 2π.



ρ varies from 0 to cos ϕ while ϕ and θ are constant.

ϕ varies from 0 to $\pi/4$ while θ is constant.

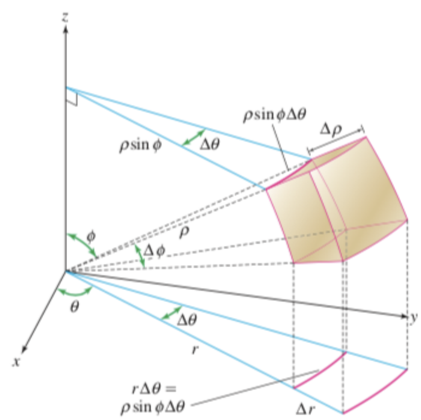
θ varies from 0 to 2π.

Therefore,

$$\begin{aligned}
\int\int\int_B \sqrt{x^2+y^2}\ dV &= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos(\phi)} \sqrt{(\rho\sin(\phi)\cos(\theta))^2+(\rho\sin(\phi)\sin(\theta))^2}\cdot\rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta \\
&= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos(\phi)} \rho\sin(\phi)\cdot\rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta \\
&= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos(\phi)} \rho^3\sin^2(\phi)\ d\rho\ d\phi\ d\theta \\
&= \frac{1}{4}\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \cos^4(\phi)\sin^2(\phi)\ d\phi\ d\theta \\
&= \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \cos^4(\phi)\sin^2(\phi)\ d\phi \\
&= \frac{\pi}{2}\cdot\{\frac{\phi}{16}+\frac{\sin(2\phi)}{64}-\frac{\sin(4\phi)}{64}-\frac{\sin(6\phi)}{192}\}\Big|_{\phi=0}^{\phi=\frac{\pi}{2}} \\
&= \frac{\pi^2}{64},
\end{aligned}$$

where in line six we have used a table of indefinite integrals.

We then briefly discussed why we use $dV=\rho^2\sin(\phi)\ d\rho d\phi d\theta$ in the conversion to spherical coordinates by looking at the diagram below.



Recall that $r=\rho\sin(\phi)$. The top face of the spherical cube above has dimensions $(\rho\sin(\phi)\Delta\theta)\times\Delta\rho$. The remaining edge (subtending an angle $\Delta\phi$) of the spherical cube is approximately $\rho\Delta\phi$. The volume of the spherical cube is approximately:

$$(\rho\sin(\phi)\Delta\theta)\cdot(\Delta\rho)\cdot(\rho\Delta\phi)=\rho^2\sin(\phi)\Delta\rho\Delta\phi\Delta\theta.$$

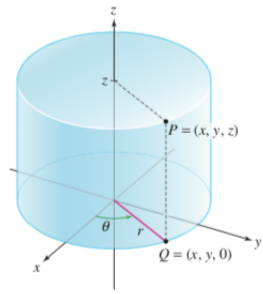
Since once can use small spherical wedges like above as a partition of B in the Riemann sum for $\int\int\int_B f(x,y,z)\ dV$, it follows that the Riemann sum in spherical coordinates takes the form

$$\sum_i\sum_j\sum_k f(\rho_i\sin(\phi_j)\cos(\theta_k),\rho_i\sin(\phi_j)\sin(\theta_k),\rho_i\cos(\phi_j))\cdot\rho^2\sin(\phi)\Delta\rho\Delta\phi\Delta\theta,$$

so that taking the limit as the volumes tend to zero gives the formula for $\int\int\int_B f(x,y,z)\ dV$ in spherical coordinates.

We then noted that every point in $\mathbb{R}^3$ lies on the top edge of a cylinder, which enables us to describe points in $\mathbb{R}^3$ in terms of cylindrical coordinates, noting that cylindrical coordinates are essentially like polar coordinates, though with the extra variable z.

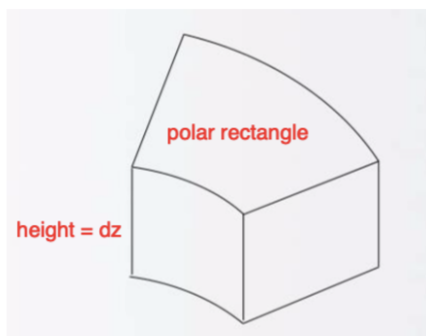
In cylindrical coordinates, $P = (r, \theta, z)$.



To transform a triple integral into cylindrical coordinates, we set:

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z, \quad dV = r\ dz\ dr\ d\theta.$$

We can easily guess what the volume of a cylindrical wedge should be.



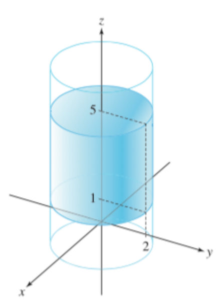
It should be approximately the area of the corresponding polar rectangle times Δz, the change in the z direction, i.e., the height of the cylindrical wedge. Thus, in cylindrical coordinates, small units of volume dV are approximately $(r\Delta r\Delta\theta)\cdot\Delta z = r\ \Delta z\ \Delta r\ \Delta\theta$. Using expressions of this type in the Riemann sum for $\int\int\int_B f(x, y, z)\ dV$ in cylindrical coordinates, we get the following version of Fubini's theorem:

Fubini's Theorem in Cylindrical Coordinates. Given a bounded region $B \subseteq \mathbb{R}^3$ described in cylindrical coordinates as: $z_1(r,\theta) \leq z_2(r,\theta); r_1(\theta) \leq r \leq r_r(\theta); \theta_1 \leq \theta \leq \theta_2$, and $f(x, y, z)$ continuous on B, we have:

$$\int\int\int_B f(x, y, z)\ dV = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)}\int_{z_1(,\theta)}^{z_2(r,\theta)} f(r\cos(\theta), r\sin(\theta), z)\ r\ dz\ dr\ d\theta.$$

We then worked some examples using cylindrical coordinates.

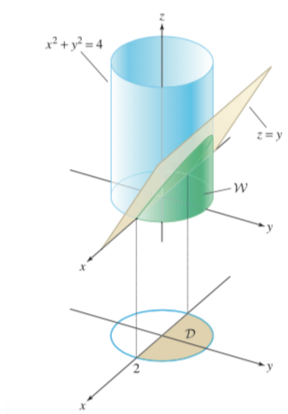
Example 2. Integrate $z\sqrt{x^2 + y^2}$ over the cylinder B given below.



Solution: In cylindrical coordinates $B: 0 \le r \le 2, 0 \le \theta \le 2\pi, 1 \le z \le 5$. Thus,

$$\begin{aligned}
\int\int\int_B z\sqrt{x^2+y^2} &= \int_0^{2\pi}\int_0^2\int_1^5 z\sqrt{(r\cos(\theta))^2+(r\sin(\theta))^2}\ r\ dzdrd\theta \\
&= \int_1^5\int_0^{2\pi}\int_0^2 zr^2\ drd\theta dz \\
&= 2\pi\int_1^5\int_0^2 zr^2\ drdz \\
&= 2\pi\int_1^5 z(\frac{2^3}{3}-0)\ dz \\
&= \frac{16\pi}{3}\int_1^5 z\ dz \\
&= \frac{16\pi}{3}\{\frac{5^2}{2}-\frac{1}{2}\} \\
&= 64\pi.
\end{aligned}$$

Example 3. Calculate $\int\int\int_W z\ dV$ for W bounded by the cylinder $0 \le x^2+y^2 \le 4$ and the planes $z = y$, with $y \ge 0$.

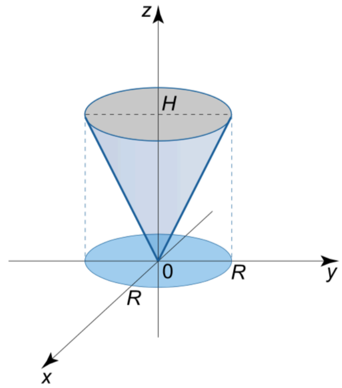


To describe W is cylindrical coordinates: $0 \le z \le y$, so $0 \le z \le r\sin(\theta)$. Since $y \ge 0$, the projection of W onto the xy-plane is the semi-circle D. Thus, $0 \le r \le 2$ and $0 \le \theta \le \pi$.

$$\begin{aligned}
\int\int\int_W z\ dV &= \int_0^{\pi}\int_0^2\int_0^{r\sin(\theta)} z\ rdzdrd\theta \\
&= \frac{1}{2}\int_0^{\pi}\int_0^2 r^3\sin^2(\theta)\ drd\theta \\
&= \frac{1}{2}\int_0^{\pi}\frac{16}{4}\sin^2(\theta)\ d\theta \\
&= 2\int_0^{\pi}\frac{1}{2}-\frac{1}{2}\cos(2\theta)\ d\theta \\
&= 2\{\frac{\theta}{2}-\frac{1}{4}\sin(2\theta)\}_0^{\pi} \\
&= \pi.
\end{aligned}$$

Example 4. Find the volume of a cone with height H and radius R.

Solution: If we take $z = \frac{H}{R}\sqrt{x^2+y^2}$, we have the cone (can you see why?):



If we let B denote the solid cone, then z is bounded below by the cone and above by the plane $z = H$. In cylindrical coordinates, the cone is $z = \frac{H}{R} \cdot r$. Thus,

$$\begin{aligned}
\text{vol}(B) &= \int\int\int_B dV \\
&= \int_0^R \int_0^{2\pi} \int_{\frac{H}{R}r}^{H} r \; dz \; dr \; d\theta \\
&= \int_0^{2\pi} \int_0^R rz\Big|_{z=\frac{H}{R}r}^{z=H} dr \; d\theta \\
&= \int_0^{2\pi} \int_0^R rH - \frac{H}{R}r^2 \; dr \; d\theta \\
&= \int_0^{2\pi} (\frac{r^2}{2}H - \frac{H}{R}\cdot\frac{r^3}{3})\Big|_{r=0}^{r=R} d\theta \\
&= \int_0^{2\pi} \frac{R^2H}{2} - \frac{R^2H}{3} \; d\theta \\
&= \frac{R^2H}{6}\int_0^{2\pi} d\theta \\
&= \frac{R^2H}{6}\cdot 2\pi \\
&= \frac{\pi R^2 H}{3}.
\end{aligned}$$

Tuesday, April 6. We worked in breakout rooms on homework problems due today at 11:59pm.

Wednesday, April 7. Today we had Quiz 7. Afterwards we set up triple integrals from our recent homework set. These problems were from Chapter 5 of the Open Stax text.

233. Solution: The key point was to insure that the plane $z + y + z = 9$ does not intersect domain in the xy-plane. The required triple integral is

$$\int_0^2 \int_{x^2+1}^{7-x} \int_0^{9-x-y} dx \; dy \; dx.$$

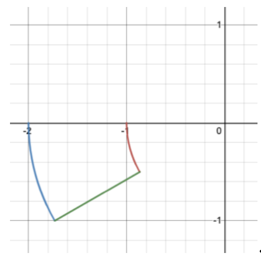
241. The required triple integral in cylindrical coordinates is

$$\int_0^{\frac{\pi}{2}} \int_0^3 \int_0^1 z \cdot r \; dz \; dr \; d\theta.$$

245. In cylindrical coordinates, the integral is

$$\int_{\pi}^{\theta} \int_{1}^{2} \int_{2}^{3} e^{r} \cdot r \; dz \; dr \; d\theta,$$

where θ is the upper bound of the polar region



.

Using that $x = \sqrt{3}y$ and $x^2 + y^2 = 1$ (say), we saw that, the intersection of the line with the circle of radius one, occurs when $x = -\frac{\sqrt{3}}{2}$ and $= -\frac{1}{2}$, so that $\theta = \frac{7\pi}{6}$.

281. We saw that the equation of the sphere can be re-written as $x^2 + y^2 + (z-1)^2 = 1$, which in spherical coordinates becomes $\rho = 2\cos(\theta)$. As in previous examples finding the volume between a sphere and a cone, we need the angle the cone makes with the z-axis. The cone is easily seen to be a 45 degree cone, so that $0 \leq \phi \leq \frac{\pi}{4}$. Thus, the required triple integral is

$$\int_{0}^{2\pi} \int_{0}^{\frac{\pi}{4}} \int_{0}^{2\cos(\phi)} \rho^2 \sin(\phi) \; d\rho \; d\phi \; d\theta.$$

In cylindrical coordinates, the cone is $z = r$ and the sphere is $z = \sqrt{1-r^2} + 1$. Setting these equations equal to each other gives $r = 1$, which means the domain of integration in the xy-plane is the unit circle centered at the origin. Thus, in cylindrical coordinates, the required integral is

$$\int_{0}^{2\pi} \int_{0}^{1} \int_{r}^{\sqrt{1-r^2}+1} r \; dz \; dr \; d\theta.$$

Thursday, April 8. We began a discussion of the change of variable principle for triple integrals. The goal here is the same as with double integrals, namely, to transform a triple integral that may be difficult, or otherwise impossible to calculate to one which is more manageable.

We have a change of variables formula for triple integrals similar to the previous one for double integrals. We start with a transformation

$$G(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))$$

from the uvw coordinate system to the xyz coordinate system. We assume:

(i) All first order partials of the coordinate functions exist and are continuous.
(ii) $G(u, v, w)$ is 1-1 on the interior of a given domain W_0 in uvw-space.
(iii) $G(B_0) = B$, where B_0 is a solid in the uvw-space and B is a solid in xyz-space.
(iv) $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$.

As before, we call $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ the Jacobian of $G(u, v, w)$, which we also denote as $\mathrm{Jac}(G)$. Here is the change of variables theorem for triple integrals.

Change of Variables Theorem. For $f(x, y, z)$ continuous on the bounded region $B \subseteq \mathbb{R}^3$, and $G(u, v, w)$ as above, $\int\int\int_B f(x, y, z) \; dV$ can be computed as:

$$\int\int\int_{B_0} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dV.$$

As in the case of the two variable change of variable formula, the formula above suggests that small units of volume dV in the x, y, z coordinate system correspond to small units of volume times $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|$ in the u, v, w coordinate system. We have already seen examples of changing variables for triple integrals, namely, spherical and cylindrical coordinates.

Spherical Coordinates. In this case, $G(\rho, \phi, \theta) = (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)$. As noted previously, $G(\rho, \phi, \theta)$ transforms the (solid) rectangular box $[0, R] \times [0, \pi] \times [0, 2\pi]$ in the ρ, ϕ, θ coordinate system to the (solid) sphere of radius R in the x, y, z coordinate system. Moreover, we have,

$$\begin{aligned}\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta))} &= \det\begin{pmatrix}\sin(\phi)\cos(\theta) & \rho\cos(\phi)\cos(\theta) & -\rho\sin(\phi)\sin(\theta)\\ \sin(\phi)\sin(\theta) & \rho\cos(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta)\\ \cos(\phi) & -\rho\sin(\phi) & 0\end{pmatrix}\\ &= \cos(\phi)\cdot(\rho^2\cos(\phi)\sin(\phi)) + \rho\sin(\phi)\cdot(\rho\sin^2(\phi))\\ &= \rho^2\sin(\phi)\cdot(\cos^2(\phi)+\sin^2(\phi))\\ &= \rho^2\sin(\phi).\end{aligned}$$

Since $\rho^2\sin(\phi) \geq 0$, for $0 \leq \phi \leq \pi$, $\left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta))}\right| = \rho^2\sin(\phi)$. This, together with the statement of the change of variables formula, explains the term $\rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$ in the formula for using spherical coordinates.

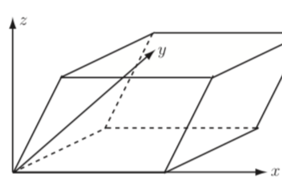
Cylindrical Coordinates. In this case, we have $G(r, \theta, z) = (r\cos(\theta), r\sin(\theta), z)$, and $G(r, \theta, z)$ transforms the (solid) rectangular box $[0, R] \times [0, 2\pi] \times [0, H]$ in the r, θ, z coordinate system to the (soild) cylinder of radius R and height H whose base is centered at the origin in the x, y, z coordinate system. Moreover,

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det\begin{pmatrix}\cos(\theta) & -r\sin(\theta) & 0\\ \sin(\theta) & r\cos(\theta) & 0\\ 0 & 0 & 1\end{pmatrix} = r.$$

Since $r \geq 0$, $\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r$. This together with the statement of the change of variables formula, explains the term $r\ dz\ rd\ d\theta$ in the formula for using cyclindrical coordinates.

We then did the following example.

Example 1. Calculate $\int\int_W (x - y)\ dV$, where B is the parallelepiped spanned by the vectors $i, j + k, i + 2k$. Here is an example of a parallelepiped (not necessarily B).



Solution. We used the linear transformation $T(u, v, w) = (u + w, v, v + 2w)$, noting that T takes the unit cube in uvw-space to B in xyz-space. This followed since, on the one hand, T is linear, so it takes straight lines to straight lines, while on the other hand, $T((1, 0, 0)) = \vec{i}, T((0, 1, 0)) = \vec{j} + \vec{k}, T((0, 0, 1)) = \vec{i} + 2\vec{k}$. We also have

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 1 & 2\end{pmatrix} = 2.$$

Therefore,

$$\begin{aligned}\int\int\int_W (x-y)\ dV &= \int_0^1\int_0^1\int_0^1 (u+w-v)\ |2|\ du\ dv\ dw\\ &= 2\int_0^1\int_0^1\int_0^1 (u-v+w)\ dudvdw\\ &= 2\int_0^1\int_0^1 \frac{1}{2}-v+w\ dvdw\\ &= 2\int_0^1 \frac{1}{2}-\frac{1}{2}+w\ dw\\ &= 2\int_0^1 w\ dw\\ &= 1.\end{aligned}$$

Example 2. A general linear transformation has the form

$$T(u,v,w) = (au+bv+cw, du+ev+gw, ru+sv+tw),$$

with $a,b,c,d,e,f,r,s,t \in \mathbb{R}$ and with Jacobian:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} a & b & c\\ d & e & g\\ r & s & t\end{pmatrix}$$

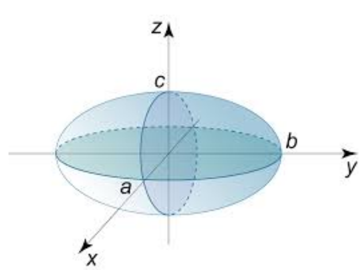
whose absolute value is the volume of the parallelepiped B spanned by the vectors $ai+dj+rk, bi+ej+sk, ci+gj+tk$.

Thus: $\int\int\int_B f(x,w,z)\ dV$ can be computed as

$$\int_0^1\int_0^1\int_0^1 f(au+bv+cw, du+ev+gw, ru+sv+tw)\left|\det\begin{pmatrix} a & b & c\\ d & e & g\\ r & s & t\end{pmatrix}\right| dudvdw,$$

since T takes the unit cube in uvw-space to B.

Example 3. Find the volume of the ellipsoid $E: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leq 1$.



Solution. We may assume $a,b,c>0$. The transformation $G(u,v,w) = (au,bv,cw) = (x,y,z)$ takes the unit sphere S in uvw-space to the ellipsoid and since $u = \frac{x}{a}, v = \frac{y}{b}, w = \frac{z}{c}$, and hence $u^2+v^2+w^2 \leq 1$. Moreover:

$\text{Jac} = \det \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = abc$. So:

$$\begin{aligned}
\text{vol}(E) &= \int\int\int_E dV \\
&= \int\int\int_S |abc| \; dudvdw \\
&= abc \int\int\int_S dV \\
&= abc \cdot \text{vol}(S) \\
&= \frac{4}{3}\pi abc.
\end{aligned}$$

Example 4. A translation $G(u, v, w) = (u + a, v + b, w + c) = (x, y, z)$ shifts the origin in the u, v, w coordinate system to the point (a, b, c) in x, y, z coordinate system. Note that

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1,$$

as expected, since G does not change volumes. Thus, for example, if B_0 is the sphere of radius R in the u, v, w coordinate system centered at the origin, then $G(B_0) = B$ is the sphere of Radius R centered at (a, b, c) in the x, y, z coordinate system. Therefore,

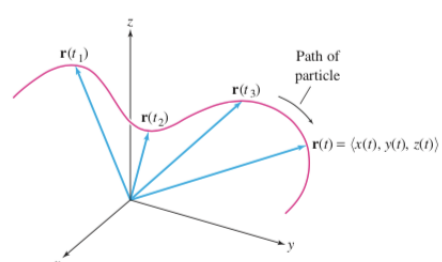
$$\int\int\int_B f(x, y, z) \; dV = \int\int\int_{B_0} f(u + a, v + b, w + c) \; dudvdw.$$

Friday, April 9. Our next goal is to integrate along a curve in $\mathbb{R}^3$. For this, we began a discussion of vector valued functions.

Definition. A vector valued function is a function

$$\mathbf{r}(t) = x(t)i + y(t)j + z(t)k = (x(t), y(t)z(t)),$$

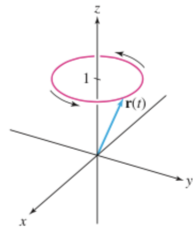
with t belonging to a subset of $\mathbb{R}$.



If the values of $\mathbf{r}(t)$ lie in the xy-plane, we write

$$\mathbf{r}(t) = (x(t), y(t)) = x(t)i + y(t)j.$$

The variable t is called the *parameter*. It is often convenient to think of t as time. The set of points traced out by $\mathbf{r}(t)$ is a *curve*. How the curve is traced out is called a *path*. For example, for the curve C, the circle of radius one, with center $(0, 0, 1)$



one can have several different paths:

$\mathbf{r}_1(t) = (\cos(t), \sin(t), 1)$, with $0 \leq t \leq 2\pi$ traces the curve once.

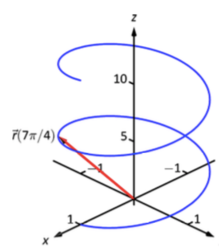
$\mathbf{r}_2(t) = (\cos(2t), \sin(2t), 1)$, with $0 \leq t \leq \pi$ traces once, but twice as fast.

$\mathbf{r}_3(t) = (\cos(t), \sin(t), 1)$, with $0 \leq t \leq 4\pi$, traces the curve twice, but at the same speed at $\mathbf{r}_1(t)$.

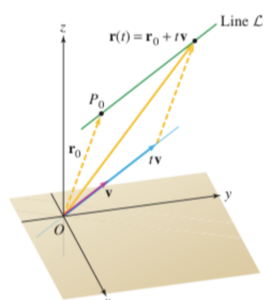
$\mathbf{r}_4(t) = (\cos(2\pi - t), \sin(2\pi - t), 1)$, traces the curve once, in the opposite direction of $\mathbf{r}_1(t)$.

The different paths $\mathbf{r}_i(t)$ above are referred to as different *parametrizations* of C. We then gave two more examples of curves in $\mathbb{R}^3$.

Example 2. The helix $\mathbf{r}(t) = (cos(t), \sin(t), t)$, $t \geq 0$.



Example 3. The line through a point $P_0 = (a, b, c)$ parallel to a given vector $\vec{v} = v_1 i + v_2 j + v_3 k$.



(B) Line through P_0 in the direction of $\mathbf{v}$

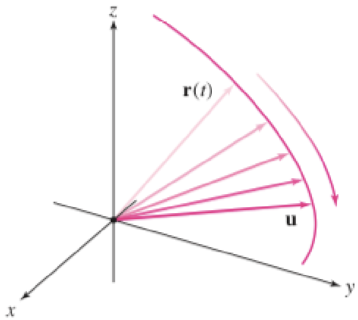
$$\mathbf{r}(t) = P_0 + t\vec{v} = (a + tv_1, b + tv_2, c + tv_3).$$

We then noted that limits and continuity for vector valued functions are defined in a similar way as for functions we have previously encountered.

Limits. For a vector valued function $\mathbf{r}(t)$ and a fixed vector $\vec{u} = u_1 i + u_2 j + u_3 k$, we write:

$$\lim_{t \to t_0} \mathbf{r}(t) = \vec{u} \quad \text{if} \quad \lim_{t \to t_0} ||\mathbf{r}(t) - \vec{u}|| = 0.$$

Note that this means the vectors $\mathbf{r}(t)$ get closer to the vector $\vec{u}$ as t approaches t_0.



DF FIGURE 1 The vector-valued function $\mathbf{r}(t)$ approaches the vector $\mathbf{u}$ as $t \to t_0$.

This is equivalent to

$$\lim_{t\to t_0} x(t) = u_1$$
$$\lim_{t\to t_0} y(t) = u_2$$
$$\lim_{t\to t_0} z(t) = u_3.$$

To explain why this holds, we noted

$$\begin{aligned}\lim_{t\to t_0} ||\mathbf{r}(t) - \vec{u}|| &= \lim_{t\to t_0} \sqrt{(x(t)-u_1)^2 + (y(t)-u_2)^2 + (z(t)-u_3)^2} \\ &= \sqrt{(\lim_{t\to t_0} x(t)-u_1)^2 + (\lim_{t\to t_0} y(t)-u_2)^2 + (\lim_{t\to t_0} z(t)-u_3)^2}\end{aligned}$$

by continuity of the square root and square functions. If

$$\lim_{t\to t_0} x(t) = u_1$$
$$\lim_{t\to t_0} y(t) = u_2$$
$$\lim_{t\to t_0} z(t) = u_3,$$

then the limits under the radical are zero. Thus, $\lim_{t\to t_0} ||\mathbf{r}(t) - \vec{u}|| = 0$. i.e., $\lim_{t\to t_0} \mathbf{r}(t) = \vec{u}$.

Example 4. If $\mathbf{r}(t) = (5\cos(t), -3\sin(\frac{t}{2}), e^{3t+4})$, then

$$\begin{aligned}\lim_{t\to\pi} \mathbf{r}(t) &= (\lim_{t\to\pi} 5\cos(t), \lim_{t\to\pi} -3\sin(\frac{t}{2}), \lim_{t\to\pi} e^{3t+4}) \\ &= (5\cos(\pi), -3\sin(\frac{\pi}{2}), e^{3\pi+4}) \\ &= (-5, -3, e^{3\pi+4}).\end{aligned}$$

Continuity. $\mathbf{r}(t)$ is *continuous at* t_0 if $\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. From the discussion of limits, we observed that this is equivalent to:

$$x(t), y(t), z(t) \text{ are all continuous at } t_0.$$

We then noted that differentiabiility is defined as expected.

Differentiability. $\mathbf{r}(t)$ is *differentiable* at t_0 if

$$\lim_{t\to h} \frac{1}{h} \cdot \{\mathbf{r}(t+h) - \mathbf{r}(t_0)\},$$

exists. Note this limit involves a scalar times a vector, and is a vector if the limit exists. If the limit exists, we write it as $\mathbf{r}'(t_0)$ or $\frac{d}{dt}\mathbf{r}(t)|_{t_0}$.

Fact: $\mathbf{r}'(t)$ is differentiable at t_0 exactly when $x(t), y(t), z(t)$ are all differentiable at t_0, in which case

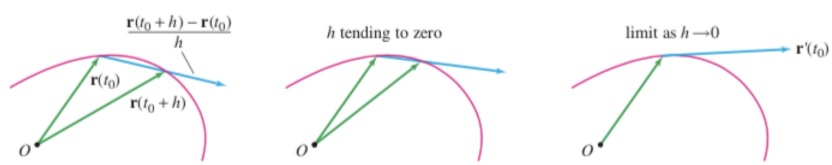
$$\mathbf{r}'(t_0) = (x'(t_0), y'(t_0), z'(t_0)).$$

This follows, since we may take limits coordinate-wise (and because the scalar $\frac{1}{h}$ can be moved inside of an ordered triple):

$$\lim_{t\to h} \frac{1}{h} \cdot \{\mathbf{r}(t+h) - \mathbf{r}(t_0)\} = (\lim_{t\to h} \frac{x(t_0+h) - x(t_0)}{h}, \lim_{t\to h} \frac{y(t_0+h) - y(t_0)}{h}, \lim_{t\to h} \frac{z(t_0+h) - z(t_0)}{h})$$
$$= (x'(t_0), y'(t_0), z'(t_0)).$$

Example 5. Given $\mathbf{r}(t) = (5\cos(t), -3\sin(\frac{t}{2}), e^{3t+4})$, $\mathbf{r}'(t) = (-5\sin(t), -\frac{3}{2}\cos(\frac{t}{2}), 3e^{3t+4})$. Therefore: $\mathbf{r}\ '(\pi) = (-5\sin(\pi), \frac{3}{2}\cos(\frac{\pi}{2}), e^{3\pi+4}) = (0, 0, e^{3\pi+4})$.

We finished class by noting that the vector $\mathbf{r}'(t_0)$ is tangent to the curve $\mathbf{r}(t)$ at the point $P_0 = \mathbf{r}(t_0)$, as illustrated in the diagram below.



Monday, April 12. We continued our discussion of vector valued functions $\mathbf{r}(t) = (x(t), y(t), z(t))$, by recalling that limits and derivatives are taken coordinate-wise and that the vector $\mathbf{r}'(t_0)$ is tangent to the curve given by $\mathbf{r}(t)$ at the point $\mathbf{r}(t_0)$. We then started with the following example

Example 1. Find the line tangent the helix $\mathbf{r}(t) = (\cos(t), \sin(t), t)$ at $(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3}) = \mathbf{r}(\frac{\pi}{3})$.

Solution. $\mathbf{r}'(\frac{\pi}{3}) = (-\sin(\frac{\pi}{3}), \cos(\frac{\pi}{3}), 1) = (-\frac{\sqrt{3}}{2}, \frac{1}{2}, 1)$. Tangent line:

$$\vec{L}(t) = (\frac{1}{2} - \frac{\sqrt{3}}{2}t, \frac{\sqrt{3}}{2} + \frac{1}{2}t, \frac{\pi}{3} + t).$$

We then noted that we have versions of the familiar rules of differentiation for vector valued functions.

Properties of the derivative. Assuming differentiablity:

(i) $(\mathbf{r}(t) + \mathbf{r}s(t))' = \mathbf{r}'(t) + \mathbf{r}s'(t)$.
(ii) $(\lambda\mathbf{r}(t))' = \lambda\mathbf{r}'(t)$, for $\lambda \in \mathbb{R}$.
(iii) $(f(t) \cdot \mathbf{r}(t))' = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$, for $f(t)$ a scalar valued function.
(iv) $\mathbf{r}(g(t))' = g'(t)\mathbf{r}'(g(t))$, for $g(t)$, a scalar function.
(v) $(\mathbf{r}(t) \cdot \vec{s}(t))' = \mathbf{r}'(t) \cdot \vec{s}(t) + \mathbf{r}(t) \cdot \vec{s}'(t)$.
(vi) $(\mathbf{r}(t) \times \vec{s}(t))' = \mathbf{r}\ '(t) \times \vec{s}(t) + \mathbf{r}(t) \times \vec{s}\ '(t)$.

Example 2. We verify formula (vi) for the functions $\mathbf{r}(t) = (t^2, 1, 2t)$ and $s(t) = (1, 2t, eT2)$. We start with

$$\mathbf{r}(t) \times \mathbf{s}(t) = \begin{vmatrix} i & j & k \\ t^2 & 1 & 2t \\ 1 & 2t & e^2 \end{vmatrix} = (e^t - 4t^2, 2t - t^2e^t, 2t^3 - 1),$$

and thus, $(\mathbf{r}(t) \times \mathbf{s}(t))' = (e^t - 8t, 2 - 2te^t - t^2e^t, 6t^2)$. On the other hand, $\mathbf{r}'(t) = (2t, 0, 2)$ and $\mathbf{s}'(t) = (0, 2, e^2)$. Therefore,

$$\mathbf{r} \times \mathbf{s}' = \begin{vmatrix} i & j & k \\ t^2 & 1 & 2t \\ 0 & 2 & e^t \end{vmatrix} = (e^t - 4t, -t^2e^2, 2t^2) \quad \text{and} \quad \mathbf{r}' \times \mathbf{s} = \begin{vmatrix} i & j & k \\ 2t & 0 & 2 \\ 1 & 2t & e^{2t} \end{vmatrix} = (-4t, -2te^t + 2, 4t^2).$$

Adding these equations we have

$$\begin{aligned}\mathbf{r} \times \mathbf{s}' + \mathbf{r}' \times \mathbf{s} &= (e^t - 4t, -t^2e^2, 2t^2) + (-4t, -2te^t + 2, 4t^2) \\ &= (e^t - 8t, 2 - 2t - t^2e^t, 6t^2) \\ &= (\mathbf{r} \times \mathbf{s})'.\end{aligned}$$

We then discussed the length of a path or curve.

Definition. Suppose $\mathbf{r}(t)$ is differentiable and $\mathbf{r}'(t)$ is continuous on $[a, b]$. Then the length of the path from $\mathbf{r}(a)$ to $\mathbf{r}(b)$ is given by

$$s = \int_a^b ||\mathbf{r}'(t)||\ dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\ dt.$$

The parameter s is called *arc length.* We can keep track of the arc length as we move along the path by considering the function:

$$s(t) = \int_a^t ||\mathbf{r}'(u)||\ du,$$

for $a \leq t \leq b$. We noted that s gives the length of the path $\mathbf{r}(t)$ for the range $a \leq t \leq b$, and that this is also the length of the corresponding curve, under an additional hypothesis.

|bf IMPORTANT POINTS. (a) If $\mathbf{r}(t)$ is 1-1, then the arc length of the path equals the length of the curve traced out by $\mathbf{r}(t)$.

(b) The length of the curve traced out by $\mathbf{r}(t)$ is *independent of the parametrization.*

Example 3. We then revisited the examples from the previous lecture of the circle of radius one centered at (0,0,1) and its four parametrizations.

(i) $\mathbf{r}_1(t) = (\cos(t), \sin(t), 1)$, with $0 \leq t \leq 2\pi$ traces the curve once.
(ii) $\mathbf{r}_2(t) = (\cos(2t), \sin(2t), 1)$, with $0 \leq t \leq \pi$ traces once, but twice as fast.
(ii) $\mathbf{r}_3(t) = (\cos(t), \sin(t), 1)$, with $0 \leq t \leq 4\pi$, traces the curve twice, but at the same speed at $\mathbf{r}_1(t)$.
(iv) $\mathbf{r}_4(t) = (\cos(2\pi - t, \sin(2\pi - t), 1)$, traces the curve, once in reverse order.

We expect the lengths of (i), (i), (iv) to be 2π and the length of the path (iii) to be 4π.

For $\mathbf{r}_1(t) : ||\mathbf{r}_1'(t)|| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 0} = 1$. Thus:

$$s = \int_0^{2\pi} ||\mathbf{r}_1'(t)||\ dt = \int_0^{2\pi} 1\ dt = 2\pi.$$

Note that $\mathbf{r}_1(t)$ is 1-1 on $[0, 2\pi]$, so this gives the expected length of the curve.

For $\mathbf{r}_2(t) : ||\mathbf{r}_2'(t)|| = \sqrt{(-2\sin(2t))^2 + (2\cos(2t))^2 + 0} = 2$. Thus:

$$s = \int_0^{\pi} ||\mathbf{r}_2'(t)||\ dt = \int_0^{\pi} 2\ dt = 2\pi.$$

Note that $\mathbf{r}_2(t)$ is also 1-1, so we get that the length of the path equals the length of the curve. Note also that the previous two parametrizations are the different, but yield the same arc length.

For $\mathbf{r}_3(t) : ||\mathbf{r}_3'(t)|| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 0} = 1$. Thus

$$s = \int_0^{4\pi} ||\mathbf{r}_3'(t)||\ dt = \int_0^{4\pi} 1\ dt = 4\pi.$$

Note here that the path traces the curve twice, so the length of the path is 4π, while the length of the curve is 2π. In this case, $\mathbf{r}_3(t)$ is NOT 1-1 on the interval $[0, 4\pi]$, which explains why the length of the path differs from the length of the curve.

For $\mathbf{r}_4(t) : ||\mathbf{r}_4'(t)|| = \sqrt{(\cos(2\pi - t))^2 + (-\sin(2\pi - t))^2 + 0^2} = 1$. Thus

$$s = \int_0^{\pi} ||\mathbf{r}_4'(t)||\ dt = \int_0^{2\pi} 1\ dt = 2\pi,$$

as expected.

We finished with the following example.

Example 4. Find the length of the curve C obtained by intersecting the sphere of radius 2 centered at the origin with the plane $z = \sqrt{2}$.

Solution. The curve is a circle on the sphere at level $z = \sqrt{2}$, parallel to the xy-plane. We need to find its radius. Note that on the sphere $z = 2\cos(\phi)$, and thus on the circle $\sqrt{2} = 2\cos(\phi)$, which gives $\cos(\phi) = \frac{\sqrt{2}}{2}$, and thus $\phi = \frac{\pi}{4}$. Thus, if we take the parametrization of the spehre, with $R = 2$ and $\phi = \frac{\pi}{4}$, we get theta parametrizing the circle as

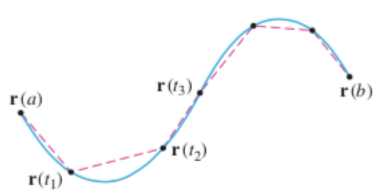
$$\mathbf{r}(\theta) = (2\sin(\frac{\pi}{4})\cos(\theta), 2\sin(\frac{\pi}{4})\sin(\theta), \sqrt{2}) = (\sqrt{2}\cos(\theta), \sqrt{2}\sin(\theta), \sqrt{2})$$

with $0 \leq \theta \leq 2\pi$. Thus

$$\begin{aligned}\text{length}(C) &= \int_0^{2\pi} ||\mathbf{r}'(\theta)||\ d\theta \\ &= \int_0^{2\pi} \sqrt{(-\sqrt{2}\sin(\theta))^2 + (\sqrt{2}\cos(\theta))^2 + 0^2}\ d\theta \\ &= \int_0^{2\pi} \sqrt{2}\ d\theta \\ &= 2\sqrt{2}\pi.\end{aligned}$$

Tuesday, April 13. We began class with Quiz 8. Afterwards, we reviewed the definition of arc length and the arc length function, noting that if we think of t as time, then $s(t) = \int_a^t ||\mathbf{r}'(t)||\ dt$ gives the distance traveled in time t along the path described by $\mathbf{r}(t)$, so that $s'(t) = ||\mathbf{r}'(t)||$ denotes the speed. Thus, the *velocity vector* $\mathbf{r}'(t)$ points in the direction of a point traveling along the curve (since it is tangent to the curve) and the length of the velocity vector gives the speed at time t.

We then gave a heuristic description accounting for the formula for the arc length along a segment of a curve, say C is given by $\mathbf{r}(t)$, with $a \leq t \leq b$. Partition the interval $[a, b] : a = t_1 < t_1 \cdots < t_n = b$ so that $t_{i+1} - t_i = \Delta t$ is small.



Create a polygonal path whose endpoints are the $\mathbf{r}(t_i)$. The length of each line segment in the path is $||\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)||$. We have:

$$||\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)|| = \sqrt{(x(t_{i+1}) - x(t_i))^2 + (y(t_{i+1}) - y(t_i))^2 + (z(t_{i+1}) - z(t_i))^2}$$

For small values of Δt:

$$\begin{aligned}x(t_{i+1}) - x(t_i) &\approx x'(t_i)\Delta t \\ y(t_{i+1}) - y(t_i) &\approx y'(t_i)\Delta t \\ z(t_{i+1}) - z(t_i) &\approx z'(t_i)\Delta t\end{aligned}$$

Thus:

$$\begin{aligned}
||\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)|| &= \sqrt{(x(t_{i+1}) - x(t_i))^2 + (y(t_{i+1}) - y(t_i))^2 + (z(t_{i+1}) - z(t_i))^2} \\
&\approx \sqrt{(x'(t_i)\Delta t)^2 + (y'(t_i)\Delta t)^2 + (z'(t_i)\Delta t)^2} \\
&= \sqrt{x'(t_i)^2 + y'(t_i)^2 + z'(t_i)^2}\ \Delta t \\
&= ||\mathbf{r}'(t_i)||\ \Delta t
\end{aligned}$$

Summing these expressions we get an approximation of the arc length on the one hand, and a Riemann sum for $\int_a^b ||\mathbf{r}'(t)||\ dt$ on the other hand. Passing to the limit as $\Delta t \to 0$ gives the arc length formula.

We then turned to the definition of $\int_C f(x, y, z)\ ds$, the line integral of the scalar function $f(x, y, z)$ along the curve C. We proceed as before, as with all previous types of integration.

Step 1. Subdivide C into finitely many smaller curves C_i of the same length Δs.

Step 2. Choose a point (x_i, y_i, z_i) from the component C_i.

Step 3. Multiply $f(x_i, y_, z_i)$ by the size of each C_i to get $f(x_i, y_i, z_i)\ \Delta s$.

Step 4. Add the products in Step 3 to get the Riemann sum: $\Sigma_i f(x_i, y_i, z_i)\Delta s$.

Step 5. Take the limit of the Riemann sums as $\Delta s \to 0$, to get:

$$\int_C f(x, y, z)\ ds,$$

the *line (or path) integral over* $f(x, y, z)$ *over* C. This integral is sometimes called the *the line integral of* $f(x, y, z)$ *with respect to arc length*.

We must use the parametrization $\mathbf{r}(t)$ of C to calculate $\int_C f(x, y, z)\ ds$. From our discussion of arc length, we have that small portions of length Δs along the curve are approximated by $||\mathbf{r}'(t_i)||\Delta t$, where $(x_i, y_i, z_i) = \mathbf{r}(t_i)$. Now use

$$\Sigma_i f(x_i, y_i, z_i)||\mathbf{r}'(t_i)||\ \Delta t$$

in the Riemann sum (Step 4) above. Taking the limit as $t \to 0$ we get:

$$\begin{aligned}
\int_C f(x, y, z)\ ds &= \int_a^b f(\mathbf{r}(t))||\mathbf{r}'(t)||\ dt \\
&= \int_a^b f(x(t), y(t), z(t))||\mathbf{r}'(t)||\ dt.
\end{aligned}$$

A word of caution: Depending upon the context, one will either calculate a *path integral*, if the parametrization is given or one may have to choose a parametrization to calculate a line integral. Moreover, if the path $\mathbf{r}(t)$ doubles back on itself, or repeats portions of the curve with non-zero length, this will be reflected in $\int_a^b f(\mathbf{r}(t))\ ||\mathbf{r}'(t)||\ dt$. Note that:

$$\int_C ds = \int_a^b ||\mathbf{r}'(t)||\ dt,$$

is the arc length of C, which is what we expect.

We finished class with two examples.

Example 1. Calculate $\int_C xe^{z^2}\ ds$, for C the line segment give by $\mathbf{r}(t) = (t, 2-t, t)$, $0 \le t \le 1$.

Solution. $\mathbf{r}'(t) = (1, -1, 1)$, so $||\mathbf{r}'(t)|| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$. Since $x = t = z$ along the curve, we have

$$\begin{aligned}\int_C xe^{z^2}\ ds &= \int_0^1 te^{t^2}\ \sqrt{3}\ dt \\ &= \sqrt{3} \cdot \frac{1}{2} e^{t^2}\Big|_{t=0}^{t=1} \\ &= \frac{\sqrt{3}}{2}(e-1).\end{aligned}$$

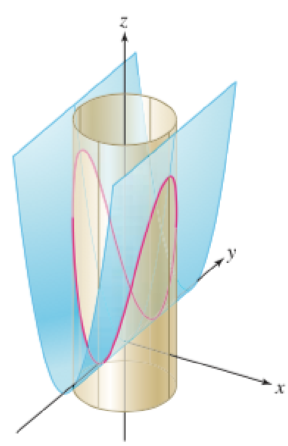
Example 2. Calculate $\int_C (x^2+y^2)z^3\ ds$ for C that portion of the helix $\mathbf{r}(t) = (\cos(t), \sin(t), t)$, with $0 \le t \le \frac{\pi}{4}$.

Solution. We have $||\mathbf{r}'(t)|| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1^2} = \sqrt{2}$. Since $x = \cos(t), y = \sin(t), z = t$ along the curve,

$$\begin{aligned}\int_C (x^2+y^2)z^3\ ds &= \int_0^{\frac{\pi}{4}} (\cos^2(t) + \sin^2(t))t^3\ \sqrt{2}\ dt \\ &= \sqrt{2}\int_0^{\frac{\pi}{4}} t^3\ dt \\ &= \frac{\sqrt{2}}{4}\ t^4\Big|_0^{\frac{\pi}{4}} \\ &= \frac{\sqrt{2}}{1024}\pi^4.\end{aligned}$$

Wednesday, April 14. We began by reviewing what a line integral of a scalar function is and how to calculate it. This was followed by some examples.

Example 1. Calculate $\int_C xy(x^2-y^2)\ ds$, where C is that portion of the curve below, lying in the first octant.



Intersection of the surfaces $x^2 + y^2 = 1$ and $z = 4x^2$.

To parametrization C, set $\mathbf{r}(t) = (\cos(t), \sin(t), 4\cos^2(t))$, with $0 \le t \le \frac{\pi}{4}$.

$$\begin{aligned}||\mathbf{r}'(t)|| &= \sqrt{(-\sin(t))^2 + (\cos(t))^2 + (8\cos(t)(-\sin(t))^2} \\ &= \sqrt{1 + 64\cos^2(t)\sin^2(t)}.\end{aligned}$$

Thus,

$$\int_C xy(x^2-y^2)\ ds = \int_0^{\frac{\pi}{4}} \cos(t)\sin(t)(\cos^2(t)-\sin^2(t))\sqrt{1+64\cos(t)^2\sin(t)^2}\ dt$$

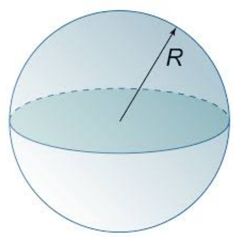
Set $u := 1+64\cos^2(t)\sin^2(t) = 64(\cos(t)\sin(t))^2$. Then:

$$du = 128\cos(t)\sin(t)(\cos^2(t)-\sin^2(t))\ dt,$$

When $t=0$, $u=1$. When $t=\frac{\pi}{4}$, $u=17$. This gives

$$\begin{aligned}\int_C xy(x^2-y^2)\ ds &= \int_0^{\frac{\pi}{4}} \cos(t)\sin(t)(\cos^2(t)-\sin^2(t))\sqrt{1+64\cos(t)^2\sin(t)^2}\ dt\\ &= \frac{1}{128}\int_1^{17}\sqrt{u}\ du\\ &= \frac{1}{192}\cdot(17^{\frac{3}{2}}-1).\end{aligned}$$

Example 2. Let C be the equator of the sphere S of radius R centered at $(0,0,0)$. Calculate $\int_C x^2+y^2+z^2\ ds$, without using a parametrization.



Solution. $2\pi R^3$. Why: On the curve $x^2+y^2+z^2=R^2$. So:

$$\int_C x^2+y^2+z^2\ ds = \int_C R^2\ ds = R^2\cdot\text{length}(C) = R^2\cdot 2\pi R = 2\pi R^3.$$

What if C is any other great circle? Same answer. What if C^* is a curve, not a great circle ? The answer depends upon C^*. One gets

$$\int_{C^*} x^2+y^2+z^2\ ds = \int_{C^*} R^2\ ds = R^2\cdot\text{length}(C^*).$$

For example, if C^* is the circle on S obtained by setting $\phi=\frac{\pi}{3}$, then using spherical coordinates, a parametrization for C^* is

$$\begin{aligned}\mathbf{r}(\theta) &= (R\sin(\frac{\pi}{3})\cos(\theta), R\sin(\frac{\pi}{3})\sin(\theta), R\cos(\frac{\pi}{3})\\ &= (\frac{\sqrt{3}R}{2}\cos(\theta), \frac{\sqrt{3}R}{2}\sin(\theta), \frac{R}{2}),\end{aligned}$$

with $0\le\theta\le 2\pi$. Thus, the length of C^* is

$$\begin{aligned}\int_0^{2\pi} ||\mathbf{r}'(\theta)||\ ds &= \int_0^{2\pi}\sqrt{(-\frac{\sqrt{3}R}{2}\sin(\theta))^2+(\frac{\sqrt{3}R}{2}\cos(\theta))^2+0^2}\ dt\\ &= \int_0^{2\pi} R\sqrt{\frac{3}{2}}\ dt\\ &= 2\pi R\sqrt{\frac{3}{2}} = \sqrt{6}\pi R.\end{aligned}$$

Thus,

$$\int_{C^*} x^2 + y^2 + z^2 \; ds = R^2 \cdot \sqrt{6}\pi R = \sqrt{6}\pi R^3.$$

One can also check this by directly computing the line integral $\int_{C^*} x^2 + y^2 + z^2 \; ds$ using the parametrization $\mathbf{r}(\theta)$ above.

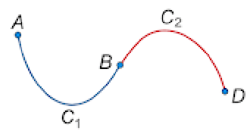
We then discussed the following properties of line integrals.

Properties of line integrals. Assuming the line integrals below exist, we have

(i) $\int_C f(x,y,z) + g(x,y,z) \; ds = \int_C f(x,y,z) \; ds + \int_C g(x,y,z) \; ds.$

(ii) $\int_C \lambda f(x,y,z) \; ds = \lambda \int_C f(x,y,z) \; ds, \; \lambda \in \mathbb{R}.$

(iii) $\int_C f(x,y,z) \; ds = \int_{C_1} f(x,y,z) \; ds + \int_{C_2} f(x,y,z) \; ds.$



(iv) $\int_C f(x,y,z) \; ds$ is independent of the parametrization, if the parametrization is 1-1.

Example 3. This example illustrates property (iii). Let C be the triangle in $\mathbb{R}^3$ with vertices (2,0,0), (0,2,0), (0,0,2). The C has three components, C_1, the line segment from (2,0,0), C_2, the line segment from (0,2,0) to (0,0,2), and C_3, then line segment from (0,0,2), to (2,0,0). By property (iii),

$$\int_C xy + z^2 \; ds = \int_{C_1} xy + z^2 \; ds + \int_{C_2} xy + z^2 \; ds + \int_{C_3} xy + z^2 \; ds.$$

The parametrizations for these curves, respectively, are:

$$\mathbf{r}_1(t) = (2 - 2t, 2t, 0), \;\; 0 \le t \le 1$$
$$\mathbf{r}_2(t) = (0, 2 - 2t, 2t), \;\; 0 \le t \le 1$$
$$\mathbf{r}_3(t) = (2t, 0, 2 - 2t), \;\; 0 \le t \le 1.$$

It is easy to check that $||\mathbf{r}_i(t)|| = 8$, for all i. Thus,

$$\int_{C_1} xy + z^2 \; ds = \int_0^1 (2-2t)2t + 0^2 \; \sqrt{8} \; dt = \frac{2\sqrt{8}}{3}$$

$$\int_{C_2} xy + z^2 \; ds = \int_0^1 0(2-2t) + (2t)^2 \; \sqrt{8} \; dt = \frac{4\sqrt{8}}{3}$$

$$\int_{C_3} xy + z^2 \; ds = \int_0^1 2t \cdot 0 + (2-2t)^2 \; \sqrt{8} \; dt = \frac{4\sqrt{8}}{3}.$$

Therefore,

$$\int_C xy + z^2 \; ds = \frac{2\sqrt{8}}{3} + \frac{4\sqrt{8}}{3} + \frac{4\sqrt{8}}{3} = \frac{10\sqrt{8}}{3}.$$

Example 4. In this example, we illustrate property (iv). Let C be the upper half of the unit circle in the xy-plane, centered at the origin. We verify that $\int_C f(x,y,z) \; ds$ is independent of the parametrization for $f(x,y) = ye^x$ and the parametrizations: $\mathbf{r}(t) = (\cos(t), \sin(t))$, $0 \le t \le \pi$ and $\mathbf{s}(t) = (\cos(\pi - 2t), \sin(\pi - 2t))$, $0 \le t \le \frac{\pi}{2}$. For $\mathbf{r}(t)$ we have

$$||\mathbf{r}'(t)|| = \sqrt{(-\sin(t))^2 + (\cos(t))^2} = 1,$$

so that

$$\int_C ye^x \, ds = \int_0^{\pi} \sin(t)e^{cos(t)} \, dt$$
$$= -e^{\cos(t)}\Big|_{t=0}^{t=\pi}$$
$$= -e^{-1} + e.$$

For the parametrization $\mathbf{s}(t)$ we have

$$||\mathbf{s}'(t)|| = \sqrt{(2\cos(\pi - 2t))^2 + (-2\sin(\pi - 2t)^2} = 2,$$

so that,

$$\int_C ye^x \, ds = \int_0^{\frac{\pi}{2}} \sin(\pi - 2t)e^{\cos(\pi-2t)} \, 2 \, dt.$$

Using u-substitution with $u = \cos(\pi - 2t)$, we have

$$\int_C ye^x \, ds = \int_0^{\frac{\pi}{2}} \sin(\pi - 2t)e^{\cos(\pi-2t)} \, 2 \, dt.$$
$$= \int_{-1}^{1} e^u \, du$$
$$= e - e^{-1},$$

which agrees with the calculation above.

We ended class with stating the following applications of a line integral.

Applications.

(i) $\frac{1}{\text{length}(C)} \int_c f(x, y, z) \, ds$ give the average value of $f(x, y, z)$ over C.

(ii) If $C \subseteq \mathbb{R}^2$ and $f(x, y) \geq 0$, for $(x, y) \in C$, then $\int_C f(x, y) \, ds$ represents the area under the surface $z = f(x, y)$ above the curve C.

(iii) If C represents a wire and $f(x, y, z)$ is the density of the wire at the point (x, y, z), then the *total mass* of the wire is $M = \int_C f(x, y, z) \, ds$.

(iv) The *center of mass* of the wire is the point $(\overline{x}, \overline{y}, \overline{z})$, where

$$\overline{x} = \frac{\int_C xf(x, y, z) \, ds}{M}, \quad \overline{y} = \frac{\int_C yf(x, y, z) \, ds}{M}, \quad \overline{z} = \frac{\int_C zf(x, y, z) \, ds}{M}.$$

Thursday, April 16. We began our discussion of surface integrals whose integrands are scalar functions. We follow the same process used in all previous forms of integration. To integrate $f(x, y, z)$ over the surface S,

Step 1. Subdivide S into finitely many smaller surfaces S_i of the same area ΔS. We are using ΔS for a small element of surface area.

Step 2. Choose a point (x_i, y_i, z_i) from the component S_i.

Step 3. Multiply $f(x_i, y_i, z_i)$ by the size of each S_i to get $f(x_i, y_i, z_i) \, \Delta S$.

Step 4. Add the products in Step 3 to get the Riemann sum: $\Sigma_i f(x_i, y_i, z_i)\Delta S$.

Step 5. Take the limit of the Riemann sums as $\Delta S \to 0$, to get:

$$\int\int_S f(x, y, z) \, dS,$$

the surface integral of $f(x, y,)$ over S. Note that we write a double integral, since our domain of integration is two-dimensional.

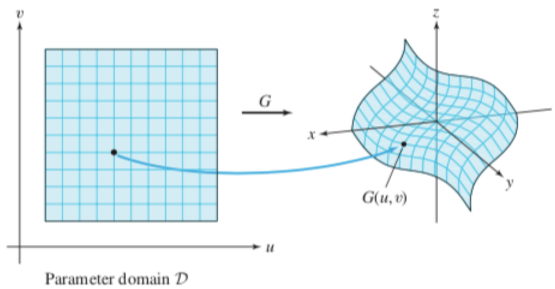
Following an analogy with curves, to calculate $\int\int_S f(x,y,z)\ dS$, we will need:

(i) A way to describe or *parametrize* a surface as a function of two variables.
(ii) A way to calculate surface area.

Definition. Given a surface $S \subseteq \mathbb{R}^3$, a parametrization of S will be a function

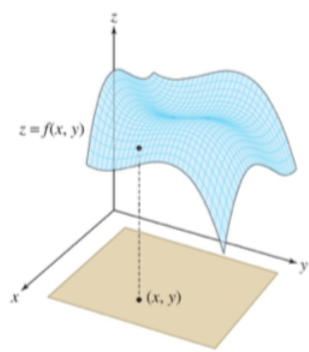
$$G(u,v) = (x(u,v), y(u,v), z(u,x)),$$

such that $S = G(D)$ for some domain D in the uv-plane.



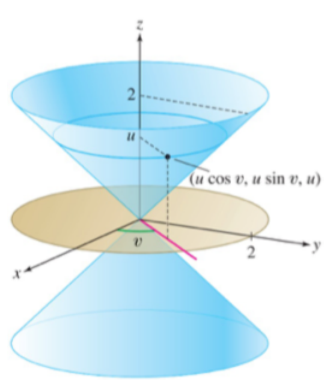
As usual, we assume that all first order partials exists and are continuous, at least on the interior of D

Example 1. The easiest surface to parametrize is a surface that is the graph of $z = f(x,y)$. Why: Because it is already defined by two parameters!



Here we can write $G(u,v) = (u, v, f(u,v))$ or $G(x,y) = (x, y, f(x,y))$. For example, if S is that portion of the paraboloid $z = x^2 + y^2$ lying over $D : 0 \leq x^2 + y^2 \leq 9$, then $G(u,v) = (u, v, u^2 + v^2)$, with $0 \leq u^2 + v^2 \leq 9$ is a parametrization that takes the disk of radius 3 in the uv-plane to S.

Example 2. Consider the cone given by $z^2 = x^2 + y^2$, with $0 \leq x^2 + y^2 \leq 4$.



Though the top half of the cone can be expressed as $z = \sqrt{x^2 + y^2}$ and we could parametrize it by $G(u, v) = (u, v, \sqrt{u^2 + v^2})$ with $0 \le x^2 + y^2 \le 4$, the whole surface cannot be expressed as the graph of a function of x and y.

Better: $G(u, v) = (u \cos(v), u \sin(v) u)$, with $-2 \le u \le 2$ and $0 \le v \le 2\pi$. Note

$$u^2 = (u \cos(v))^2 + (u \sin(v))^2,$$

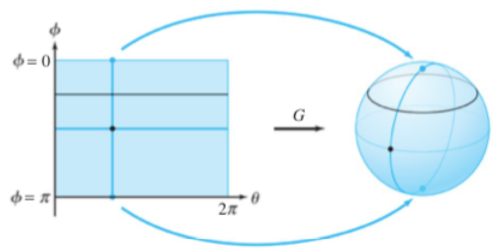
so the points $G(u, v)$ lie on the cone. More over, if we hold u fixed at u_0 and let v vary between 0 and 2π, this vertical line segment in the uv-plane is taken by G to the circle $G(u_0, v) = (u_0 \cos(v), u_0 \sin(v), u_0)$, i.e., the circle of radius u_0 centered at $(0, 0, u_0)$. As u_0 varies between 2 and -2, these circles sweep out the cone.

Spherical and cylindrical coordinates tell us how to parametrize spheres and cylinders.

Example 3. For S the sphere of radius R centered at the origin we take:

$$G(\phi, \theta) = (R \sin(\phi) \cos(\theta), R \sin(\phi) \sin(\theta), R \cos(\phi)),$$

with $0 \le \phi \le \pi, 0 \le \theta < 2\pi$.

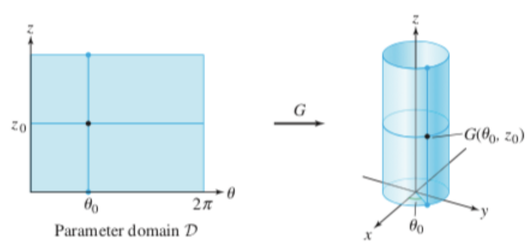


As we have seen in our earlier discussion concerning spherical coordinates, this transformation take the vertical line segment $\phi = \phi_0$, with $0 \le \theta \le 2\pi$ in the $\phi\theta$-plane to the circle $(R \sin(\phi_0) \cos(\theta), R \sin(\phi_0) \cos(\theta), R \cos(\phi_0))$ of radius $R \sin(\phi_0)$ centered at $(0, 0, R \cos(\phi_0))$. As ϕ_0 varies from 0 to π these circles sweep our the sphere.

Example 4. For the cylinder of radius R bounded below by $z = c$ and above by $z = d$, we have

$$G(u, v) = (R \cos(\theta), R \sin(\theta), z),$$

with $0 \le \theta < 2\pi$, $c \le z \le d$.



Note that if we hold z fixed at z_0, then the horizontal line segment with $0 \le \theta \le 2\pi$ in the θz-plane gets taken to the circle $(R \cos(\theta), R \sin(\theta), z_0)$ of radius R with center $(0, 0, z_0)$. As z varies between c and d, these circles sweep out the cylinder.

Now that we have a description of a surface in terms of parameters u, v, we can use this parameterization to find the plane tangent to the surface at $P = (x_0, y_0, z_0) = G(u_0, v_0)$. Since the equation of any plane is determined by a point on the plane and a vector normal to the plane, to find the tangent plane to the surface S at a point P, we find two tangent vectors, and take their cross product to find a normal vector.

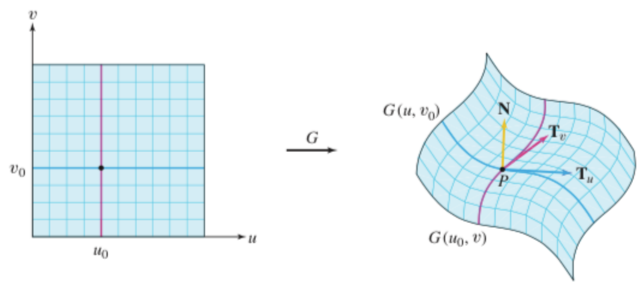
Suppose $P = G(u_0, v_0)$. How do we get tangent vectors to S at P? Hold v fixed at v_0 and let u vary. Then $G(u, v_0)$ gives a curve C_1 that moves in the u-direction on S passing through P.

Thus, $\mathbf{T}_u(u_0,v_0) = \frac{\partial G}{\partial u}(u_0,v_0)$ is a vector tangent to C_1, and hence S, at P.

$$\mathbf{T}_u(u_0,v_0) = \frac{\partial x}{\partial u}(u_0,v_0)i + \frac{\partial y}{\partial u}(u_0,v_0)j + \frac{\partial z}{\partial u}(u_0,v_0)k.$$

We get a second tangent vector to S by taking a tangent to the curve obtained by holding u fixed at u_0.

$$\mathbf{T}_v(u_0,v_0) = \frac{\partial x}{\partial v}(u_0,v_0)i + \frac{\partial y}{\partial v}(u_0,v_0)j + \frac{\partial z}{\partial v}(u_0,v_0)k.$$



The normal vector to S at P is

$$\begin{aligned}\mathbf{N} =& \mathbf{T}_u(u_0,v_0) \times \mathbf{T}_v(u_0,v_0),\\ =& \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u}(u_0,v_0) & \frac{\partial y}{\partial u}(u_0,v_0) & \frac{\partial z}{\partial u}(u_0,v_0) \\ \frac{\partial x}{\partial v}(u_0,v_0) & \frac{\partial y}{\partial v}(u_0,v_0) & \frac{\partial z}{\partial v}(u_0,v_0) \end{vmatrix}.\end{aligned}$$

With this information, we can then find the equation of the plane tangent to S at the point $P = G(u_0,v_0)$.

Friday, April 16. After recalling surface parameterizations, and the associated tangent and normal vectors we worked the following example showing how to find planes tangent to a surface.

Example 1. Find the equation of the tangent plane to the sphere of radius 2 centered at the origin at the point $P = (1,1,\sqrt{2})$.

Solution. $G(\phi,\theta) = (2\sin(\phi)\cos(\theta), 2\sin(\phi)\sin(\theta), 2\cos(\phi))$. $G(\frac{\pi}{4},\frac{\pi}{4}) = (1,1,\sqrt{2})$. We have:

$$\mathbf{T}_\phi = (2\cos(\phi)\cos(\theta), 2\cos(\phi)\sin(\theta), -2\sin(\phi)) \quad \text{and} \quad \mathbf{T}_\theta = (-2\sin(\phi)\sin(\theta), 2\sin(\phi)\cos(\theta), 0).$$

Thus, at the point P, we have

$$\mathbf{T}_\phi(\frac{\pi}{4},\frac{\pi}{4}) = (1,1,-\sqrt{2}) \quad \text{and} \quad \mathbf{T}_\theta(\frac{\pi}{4},\frac{\pi}{4}) = (-1,1,0),$$

so

$$\mathbf{T}_\phi(\frac{\pi}{4},\frac{\pi}{4}) \times \mathbf{T}_\theta(\frac{\pi}{4},\frac{\pi}{4}) = \begin{vmatrix} i & j & k \\ 1 & 1 & -\sqrt{2} \\ -1 & 1 & 0 \end{vmatrix} = \sqrt{2}i + \sqrt{2}j + 2k.$$

Therefore, the tangent plane at P is:

$$\sqrt{2}(x-1) + \sqrt{2}(y-1) + 2(z-\sqrt{2}) = 0.$$

We then noted that the process for finding a tangent plane to a parametrized surface a yields the same result for tangent planes given in the lecture of February 10. This follows, since if the surface is given by $z = f(x,y)$, then $G(x,y) = (x,y,f(x,y))$ is a parametrization. Thus, if we want the tangent vectors at the point $(a,b,f(a,b))$ using $G(x,y)$, we have $\mathbf{T}_x(a,b) = (1,0,\frac{\partial f}{\partial x}(a,b))$ and $\mathbf{T}_y(a,b) = (0,1,\frac{\partial f}{\partial y}(a,b))$, which are precisely the same tangent vectors as before.

Given the parametrized surface, why is $\mathbf{T}_u(u_0,v_0)$ tangent to the surface at $P = (x_0,y_0,z_0) = G(u_0,v_0)$? If we hold v fixed at v_0 then $\mathbf{r}(u) = (x(u,v_0), y(u,v_0), z(u,v_0))$ is a curve on the surface passing through P. Thus, $\mathbf{r}'(u_0)$ is tangent to the curve, and hence the surface, at $P = \mathbf{r}(u_0)$. However, $\mathbf{r}'(u_0) = \mathbf{T}_u(u_0,v_0)$, which shows the latter vector is tangent to the surface at P. Similarly, $\mathbf{T}_v(u_0,v_0)$ is also tangent to the surface at P. It follows that, as long as $\mathbf{N} = \mathbf{T}_u(u_0, v_0) \times \mathbf{T}_v(u_0, v_0) \neq \mathbf{0}$, we can use $\mathbf{N}$ to calculate the tangent plane to the surface at the point P.

Formula for surface area. If the bounded surface S is given by $G(u, v)$, with $G(D) = S$, for D in the uv-planes, then:

$$\text{surface area}(S) = \int\int_D ||\mathbf{T}_u \times \mathbf{T}_v||\ dA,$$

where the double integral is a standard double integral in the uv-plane.

Example 2. Find the surface area of a sphere S of radius R.

Solution. We use

$$G(\phi, \theta) = (R\sin(\phi)\cos(\theta), R\sin(\phi)\sin(\theta), R\cos(\phi)),$$

with $0 \leq \phi \leq \pi, 0 \leq \theta < 2\pi$, which defines D.

$$\mathbf{T}_\phi = (R\cos(\phi)\cos(\theta), R\cos(\phi)\sin(\theta), -R\sin(\phi)) \quad \text{and} \quad \mathbf{T}_\theta = (-R\sin(\phi)\sin(\theta), R\sin(\phi)\cos(\theta), 0).$$

Thus,

$$\begin{aligned} \mathbf{T}_\phi \times \mathbf{T}_\theta &= \begin{vmatrix} i & j & k \\ R\cos(\phi)\cos(\theta) & R\cos(\phi)\sin(\theta) & -R\sin(\phi) \\ -R\sin(\phi)\sin(\theta) & R\sin(\phi)\cos(\theta) & 0 \end{vmatrix} \\ &= R^2\sin(\phi)(\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)) \end{aligned}$$

We have:

$$||\mathbf{T}_\phi \times \mathbf{T}_\theta|| = R^2\sin(\phi),$$

since $(\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))$ lies on the sphere of radius 1. Note that $R^2\sin(\phi)$ is non-negative since $0 \leq \phi \leq \pi$. Thus:

$$\begin{aligned} \text{surface area}(S) &= \int\int_D ||\mathbf{T}_u \times \mathbf{T}_v||\ dA \\ &= \int_0^{2\pi}\int_0^{\pi} R^2\sin(\phi)\ d\phi d\theta \\ &= 2\pi R^2 \int_0^{\pi} \sin(\phi)\ d\phi \\ &= 2\pi R^2(-\cos(\phi))\Big|_0^{\pi} \\ &= 4\pi R^2. \end{aligned}$$

Question: Do you see a connection between the formula for the volume of a sphere of radius R and its surface area?

Answer: The surface area is the derivative of the volume.

We will justify the formula for surface area in the next lecture. For now, we will see that retracing our steps in defining a surface integral using a parameterization leads to a formula for calculating $\int\int_S f(x, y, z)\ dS$.

Step 1. Subdivide S into finitely many smaller surfaces S_i of area $\Delta S \approx ||\mathbf{T}_u \times \mathbf{T}_v||\ dA$.

Step 2. Choose a point $G(u_i, v_i)$ from the component S_i.

Step 3. Multiply $f(G(u_i, v_i))$ the area of S_i to get

$$f(G(u_i, v_i)) \cdot ||(\mathbf{T}_u \times \mathbf{T}_v)(u_i, v_i)||\ dA.$$

Step 4. Add the products in Step 3 to get the Riemann sum :

$$\Sigma_i f(G(u_i, v_i)) \cdot ||(\mathbf{T}_u \times \mathbf{T}_v)(u_i, v_i)||\ dA.$$

Step 5. Take the limit of the Riemann sums as $\Delta S \to 0$, to get:

$$\int\int_S f(x,y,z)\ dS = \int\int_D f(G(u_i,v_i)) \cdot ||(\mathbf{T}_u \times \mathbf{T}_v||\ dA$$
$$= \int\int_D f((x(u,v), y(u,v), z(u,v)) \cdot ||(\mathbf{T}_u \times \mathbf{T}_v||\ dudv.$$

Example 4. Calculate $\int\int_S \frac{1}{1+4(x^2+y^2)}\ dS$ for S the paraboloid $z = x^2+y^2$, $0 \le z \le 4$.

Solution. We take $G(u,v) = (u, v, u^2+v^2)$, with $0 \le u^2+v^2 \le 4$. We need to calculate $||\mathbf{T}_u \times \mathbf{T}_v||$.

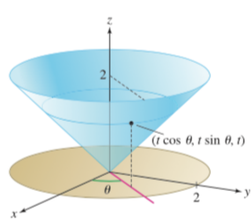
$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2ui - 2vj + k.$$

Thus $||\mathbf{T}_u \times \mathbf{T}_v|| = \sqrt{(-2u)^2 + (-2v)^2 + 1^2} = \sqrt{4u^2+4v^2+1}$. Using this in the formula for surface integrals, we have

$$\int\int_S \frac{1}{1+4(x^2+y^2)}\ dS = \int\int_D \frac{1}{1+4(u^2+v^2)} \cdot \sqrt{1+4u^2+4v^2}\ dA$$
$$= \int\int_D (1+4u^2+4v^2)^{-\frac{1}{2}}\ dudv$$
$$= \int_0^{2\pi}\int_0^2 (1+4(r\cos(\theta))^2 + 4(r\sin(\theta))^2)^{-\frac{1}{2}}\ rdrd\theta$$
$$= 2\pi \int_0^2 (1+4r^2)^{-\frac{1}{2}}\ rdr$$
$$= 2\pi \cdot \frac{1}{4}(1+4r^2)^{\frac{1}{2}}\Big|_0^2$$
$$= \frac{\pi}{2}\{\sqrt{17}-1\}.$$

Monday, April 19. We continued our discussion of surface integrals, beginning with the following examples.

Example 1. Calculate $\int\int_S x^2 z\ dS$, where S is that portion of the cone $S: z^2 = x^2+y^2$ lying above the disk $D: 0 \le x^2+y^2 \le 4$.



Solution. From last lecture, $G(u,v) = (u\cos(v), u\sin(v), u)$, with $0 \le u \le 2$, $0 \le v \le 2\pi$, is a parametrization of S.

$\mathbf{T}_u = (\cos(v), \sin(v), 1)$ and $\mathbf{T}_v = (-u\sin(v), u\cos(v), 0)$.

$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} i & j & k \\ \cos(v) & \sin(v) & 1 \\ -u\sin(v) & u\cos(v) & 0 \end{vmatrix}$$
$$= (-u\cos(v), -u\sin(v), u)$$

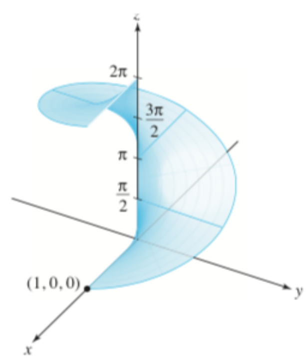
$$||\mathbf{T}_u \times \mathbf{T}_v|| = \sqrt{(-u\cos(v))^2 + (-u\cos(v))^2 + u^2} = \sqrt{2u^2} = \sqrt{2}|u|.$$

$$\int\int_S x^2 z\ dS = \int\int_D (ucos(v))^2 u \cdot \sqrt{2}|u|\ dudv$$
$$= \sqrt{2}\int_0^{2\pi}\int_0^2 u^4\cos^2(v)\ dudv,$$

since u is non-negative on D.

$$= \sqrt{2}\cdot\frac{32}{5}\int_0^{2\pi}\cos^2(v)\ dv$$
$$= \frac{32\sqrt{2}}{5}\int_0^{2\pi}\frac{1}{2}+\frac{1}{2}\cos(2v)\ dv$$
$$= \frac{32\sqrt{2}}{5}(\frac{v}{2}+\frac{1}{4}\sin(2v))\Big|_0^{2\pi}$$
$$= \frac{32\sqrt{2}}{5}\pi$$

Example 2. Calculate $\int\int_S \sqrt{x^2+y^2+1}\ dS$ where S is the helicoid: $G(r,\theta) = (r\cos(\theta), r\sin(\theta), \theta)$, with $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.



Solution. $\mathbf{T}_r = (\cos(\theta), \sin(\theta), 0)$ and $\mathbf{T}_\theta = (-r\sin(\theta), r\cos(\theta), 1)$.

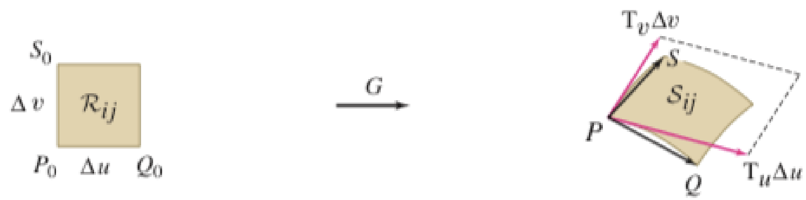
$$\mathbf{T}_r \times \mathbf{T}_\theta = \begin{vmatrix} i & j & k \\ \cos(\theta) & \sin(\theta) & 0 \\ -r\sin(\theta) & r\cos(\theta) & 1 \end{vmatrix}$$
$$= (\sin(\theta), -\cos(\theta), r).$$

$||\mathbf{T}_r \times \mathbf{T}_\theta|| = \sqrt{\sin^2(\theta) + (-\cos(\theta))^2 + r^2} = \sqrt{1+r^2}$.

$$\int\int_S \sqrt{x^2+y^2+1}\ dS = \int_0^{2\pi}\int_0^1 \sqrt{(r\cos(\theta))^2 + (r\sin(\theta))^2 + 1}\cdot\sqrt{r^2+1}\ drd\theta$$
$$= \int_0^{2\pi}\int_0^1 \sqrt{r^2+1}\cdot\sqrt{r^2+1}\ drd\theta$$
$$= 2\pi\int_0^1 r^2+1\ dr$$
$$= 2\pi\cdot\frac{4}{3}$$
$$= \frac{8\pi}{3}.$$

In order to understand our formulas for calculating surface area and surface integrals, we need to understand, where does the formula for surface area come from? For this we will use the fact from Calculus 2 that if $\mathbf{v}_1 = ai + bj + ck$ and $\mathbf{v}_2 = di + cj + ek$, then the area of the parallelogram spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ is $||\mathbf{v}_1 \times \mathbf{v}_2||$.

We first subdivide S into small portions, S_i with surface area ΔS. We approximate the small portions of surface area ΔS with small approximating tangent parallelograms. We start with the parametrization $G(u, v)$ of S. We use a tangent parallelogram to estimate the surface of the curved parallelogram on the surface.



Note that the vector from P to Q is $G(u_0 + \Delta u, v_0) - G(u_0, v_0)$. This vector is approximated by the tangent vector in red, $\Delta u \cdot \mathbf{T}_u(u_0, v_0)$. Similarly, the other tangent vector in red $\Delta v \cdot \mathbf{T}_v(u_0, v_0)$ approximates the vector $G(u_0, v_0 + \Delta v) - G(u_0, v_0)$. Thus, our small approximating tangent parallelogram is spanned by the vectors $\Delta u \cdot \mathbf{T}_u(u_0, v_0)$ and $\Delta v \cdot \mathbf{T}_v(u_0, v_0)$. The area of the approximating rectangle is

$$||(\Delta u \cdot \mathbf{T}_u) \times (\Delta v \cdot \mathbf{T}_v)|| = ||\mathbf{T_u}(u_0, v_0) \times \mathbf{T}_v(u_0, v_0)||\Delta u \Delta v.$$

Adding these areas for each component of the subdivision and taking the limit as $\Delta u, \Delta v \to 0$, gives a total surface area of $\int\int_D ||\mathbf{T}_u \times \mathbf{T}_v||\ dA$.

We then recorded the following familiar looking properties for surface integrals of continuous functions defined on a surface S:

(i) $\int\int_S f + f\ dS = \int\int_S f\ dS + \int\int_S g\ dS$.
(ii) $\int\int_S \lambda f\ dS = \lambda \int\int_S f\ dS$, for $\lambda \in \mathbb{R}$.
(iii) If $S = S_1 \cup S_2$, then $\int\int_S f\ dS = \int\int_{S_1} f]dS + \int\int_{S_2} f\ dS$, as long as S_1 and S_2 only intersect along their boundaries.
(iv) Surface area$(S) = \int\int_S\ dS$.

Important comments about calculating surface area and surface integrals.

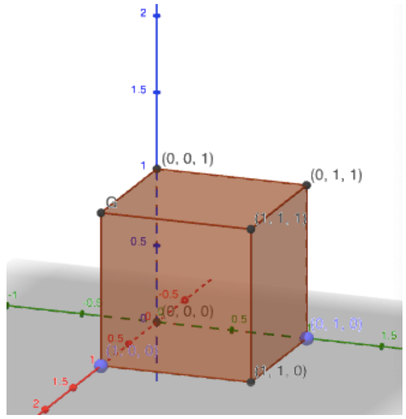
1. The quantity $||\mathbf{T}_u \times \mathbf{T}_v||$ should always be non-negative (and almost never zero). If you you calculate this expression and it involves variables from the parametrization, this function should be positive on the domain of integration. For example, if you get $||\mathbf{T}_u \times \mathbf{T}_v|| = 2uv$, then you have to make sure that the product uv is positive on the domain of integration. If not, you may have dropped an absolute value somewhere.

2. Aside from having component functions with continuous first order partial derivatives, $G(u, v)$ should be 1-1 on the interior of the domain D. Otherwise, one may be double counting portions of the surface area.

3. For calculating surface area and surface integrals of scalar functions, the orientation of the normal vector does not matter, because these quantities use the length of the normal vector. However, it will matter when we integrate vector valued functions over a surface. After all, we care whether or not a fluid flows into or out of a chamber. Or as Hagrid might say: Better out than in.

We ended class by working the following examples, showing how to use Property (iii) above.

Example 3. Calculate $\int\int_S xyz\ dS$, where S is the union of the six faces of the unit cube in $\mathbb{R}^3$.



Solution. Let $S_1, S_2, S_3, S_4, S_5, S_6$ respectively denote the bottom, left, back, right, front, and top faces of S, so that

$$\int\int_S xyz\ dS = \int\int_{S_1} xyz\ dS + \cdots + \int\int_{S_6} xyz\ dS.$$

Note that on $z = 0$ on S_1, and thus $xyz = 0$ on S_1, which gives $\int\int_{S_1} xyz\ dS = 0$. Similarly, $y = 0$ on S_2, so that $\int\int_{S_2} xyz\ dS = 0$ and $x = 0$ on S_3, so that $\int\int_{S_3} xyz\ dS = 0$.

We may parametrize the remaining sides of S as follows:

(i) $S_4 : G(u, v) = (u, 1, v)$ with $0 \leq u, v \leq 1$.
(ii) $S_5 : G(u, v) = (1, u, v)$, with $0 \leq u, v \leq 1$.
(iii) $S_6 : G(u, v) = (u, v, 1)$, with $0 \leq u, v \leq 1$.

Note that on each of these surfaces, $xyz = uv$. For $S_4 : \mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -j$, so that $||\mathbf{T}_u \times \mathbf{T}_v|| = 1$.

Thus,

$$\begin{aligned} \int\int_{S_4} xyz\ dS &= \int_0^1 \int_0^1 uv\ 1 \cdot du\ dv \\ &= \frac{1}{2}\int_0^1 v\ dv \\ &= \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{1}{4}. \end{aligned}$$

It is easy to check that for S_5, $\mathbf{T}_u \times \mathbf{T}_v = i$, so that $||\mathbf{T}_u \times \mathbf{T}_v|| = 1$. Thus,

$$\int\int_{S_5} xyz\ dS = \int_0^1 \int_0^1 uv\ dv\ dv = \frac{1}{4}.$$

Likewise,

$$\int\int_{S_6} xyz\ dS = \int_0^1 \int_0^1 uv\ dv\ dv = \frac{1}{4}.$$

Thus,

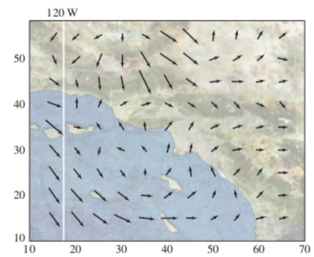
$$\int\int_S xyz\ dS = 0 + 0 + 0 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

Tuesday, April 20. We started class with Quiz 9 and then began our discussion of vector fields.

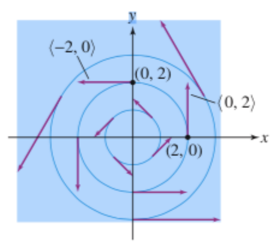
Definition. A vector field is a vector valued function

$$\mathbf{F}(x,y,z) = F_1(x,y,z)i + F_2(x,y,z,)j + F_3(x,y,z)k,$$

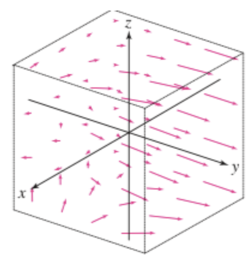
that assigns a vector to each point in a subset $S \subseteq \mathbb{R}^3$. Here's a picture of wind speed and wind direction as a vector field.



We will make our usual assumptions that the first order partial derivatives of all component functions exist and are continuous in suitable regions contained in $\mathbb{R}^3$. Some other pictures of vector fields:



$\mathbf{F} = \langle -y, x \rangle$



(A) $\mathbf{F} = \langle x \sin z, y^2, x/(z^2+1) \rangle$

Tangent vector fields and normals vector fields play a central role in what we want to do next.

Case 1. Let C be a smooth curve. Then at each point $P = (x,y,z)$ of the curve, we can assign a tangent vector $\mathbf{F}(x,y,z)$ to C at the point P. This gives a vector field along the curve C. Note that if $\mathbf{r}(t)$ is a parametrization of C, then $\mathbf{r}'(t)$ is a tangent vector at each point along the curve. Recall that $\mathbf{r}'(t)$ points in the direction of a point traveling along the curve and the length of $\mathbf{r}'(t)$ gives the speed of the point at time t.

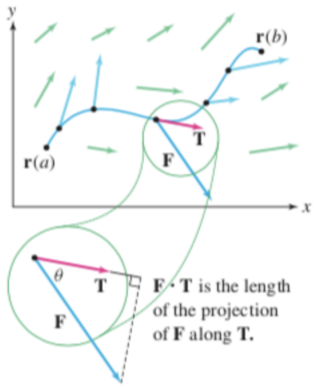
If we want to just keep track of the direction of a point moving along the curve, we can consider the unit tangent vector $\mathbf{T}(x,y,z)$ along the curve. If $\mathbf{r}(t)$ is a parametrization of C and $\mathbf{r}'(t) \neq 0$, then

$$\mathbf{T}(x,y,z) = \frac{1}{||\mathbf{r}'(t)||} \cdot \mathbf{r}'(t).$$

How do we use the unit tangent vectors? If we have a vector field $\mathbf{F}$ pushing a point along a curve, the dot product $\mathbf{F}(P) \cdot \mathbf{T}(P)$ gives us the component of $\mathbf{F}$ in the direction of the curve since:

$$\mathbf{F}(P) \cdot \mathbf{T}(P) = ||\mathbf{F}(P)|| \cdot ||\mathbf{T}(P)|| \cos(\theta) = ||\mathbf{F}(P)|| \cos(\theta),$$

where θ is the angle between $\mathbf{F}(P)$ and $\mathbf{T}(P)$.



Informally, we can think of $\mathbf{F} \cdot \mathbf{T}$ as "how much of $\mathbf{F}$ acts in the direction of $\mathbf{T}$". We will see that when we want to integrate a vector field $\mathbf{F}$ along the curve C, we will really be integrating the scalar function $\mathbf{F} \cdot \mathbf{T}$ along the C.

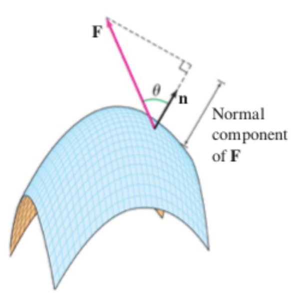
Case 2. If we have a smooth surface S, then the normal vector $\mathbf{N}$ at each point of S defines a vector field on S.

Wednesday, April 21. We began by reviewing the notion of vector field and recalling that given a curve C, the unit tangent vector $\mathbf{T}$ at each point on C defines a vector field on C, and that for a general vector field $\mathbf{F}$, $\mathbf{F} \cdot \mathbf{T}$ given the component of $\mathbb{F}_3$ along the vector $\mathbf{T}$. We then considered the case of normal vectors to a surface S.

Let S be a smooth surface in $\mathbb{R}^3$. We obtain a vector field $\mathbf{F}(x, y, z)$ by assigning to each point P on the surface a vector that is normal to S at P. At each point there are two normal vectors, each the negative of the other. The parametrization of the surface may not always yield the desired vector. If $G(u, v)$ is a parametrization of S, then $\mathbf{N}(u, v) = \mathbf{T}_u \times \mathbf{T}_v$ gives a normal vector and

$$\mathbf{n}(u, v) = \frac{1}{||\mathbf{T}_u(u, v) \times \mathbf{T}(u, v)_v||} \cdot \mathbf{N}(\mathbf{u}, \mathbf{v}),$$

gives a unit normal to the surface. If we have a vector field $\mathbf{F}$ passing through a surface - think of a fluid passing through a membrane - then $\mathbf{F}(P) \cdot \mathbf{n}(P)$ gives the component of $\mathbf{F}(P)$ in the direction of $\mathbf{n}(P)$. In fluid mechanics, one would call this the flux of the fluid across the boundary S at the point P.



We can now define line and surface integrals of vector fields.

Definition. Given a curve $C : \mathbf{r}(t)$ and a vector field $\mathbf{F}$, the line integral of $\mathbf{F}$ along C is the line integral of the scalar function $\mathbf{F} \cdot \mathbf{T}$, and is denoted $\int_C \mathbf{F} \cdot d\mathbf{r}$. Here $\mathbf{T}$ is the unit tangent along the curve pointing in the direction we travel along the curve. In other words,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds,$$

where the quantity on the right is the line integral along C of the scalar function $\mathbf{F}\cdot\mathbf{T}$. In terms of calculating the line integral, given the parametrization $\mathbf{r}(t)$ we have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C (\mathbf{F}\cdot\mathbf{T})\ ds \\ &= \int_a^b \{\mathbf{F}(\mathbf{r}(t))\cdot(\frac{1}{||\mathbf{r}'(t)||}\mathbf{r}'(t))\}\ ||\mathbf{r}'(t)||dt \\ &= \int_a^b \mathbf{F}(x(t),y(t),z(t))\cdot\mathbf{r}'(t)\ dt,\end{aligned}$$

since $\frac{1}{||\mathbf{r}'(t)||}$ cancels with $||\mathbf{r}'(t)||$. Thus, we do not have to calculate $||\mathbf{r}'(t)||$ even though it is implicit in the definition of $\int_C \mathbf{F}\cdot d\mathbf{r}$. Note that to calculate $\mathbf{F}(\mathbf{r}(t)) = \mathbf{F}(x(t),y(t),z(t))$, everywhere x, y, or z appear in the formula for $\mathbf{F}$ we replace these by $x(t), y(t), z(t)$ given in the parametrization of C.

Example 1. Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$, for C given by $\mathbf{r}(t) = (t+1, e^t, t^2),\ \ 0 \le t \le 2$ and $\mathbf{F} = zi + y^2 j + xk$.

Solution. $\mathbf{F}(\mathbf{r}(t)) = t^2 i + (e^t)^2 j + (t+1)k$ and $\mathbf{r}'(t) = i + e^t j + 2tk$. Thus,

$$\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = t^2 + e^{3t} + 2t(t+1) = e^{3t} + 2t + 3t^2.$$

Therefore,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^2 e^{3t} + 2t + 3t^2\ dt = \frac{1}{3}(e^6 + 35).$$

Definition. Given an surface S parametrized by $G(u,v) = (x(u,v), y(u,v), z(u,v))$, and a vector field $\mathbf{F}$, the surface integral of $\mathbf{F}$ over S is the surface integral of the scalar function $\mathbf{F}\cdot\mathbf{n}$, and is denoted $\int\int_S \mathbf{F}\cdot d\mathbf{S}$. Here, $\mathbf{n}$ is the unit normal on the surface determined by the parametrization, i.e., $\mathbf{n} = \frac{1}{||\mathbf{T}_u\times\mathbf{T}_v||}\mathbf{T}_u\times\mathbf{T}_v$

In other words, by definition,

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} = \int\int_S (\mathbf{F}\cdot\mathbf{n})\ dS.$$

We calculate $\int\int_S \mathbf{F}\cdot d\mathbf{S}$ using a parametrization. If

$$G(u,v) = (x(u,v), y(u,v), z(u,v)),\ \text{for } (u,v)\in D$$

is a parametrization of S, the surface integral of $\mathbf{F}$ over S is given by

$$\begin{aligned}\int\int_S \mathbf{F}\cdot d\mathbf{S} &= \int\int_S \mathbf{F}\cdot\mathbf{n}\ dS \\ &= \int\int_D \mathbf{F}(G(u,v))\cdot\{\frac{1}{||\mathbf{T}_u\times\mathbf{T}_v||}\mathbf{T}_u\times\mathbf{T}_v\}\ ||\mathbf{T}_u\times\mathbf{T}_v||\ dA \\ &= \int\int_D \mathbf{F}(x(u,v),y(u,v),z(u,v))\cdot\mathbf{T}_u\times\mathbf{T}_v\ dA.\end{aligned}$$

Thus, in a similar vein to the line integral above, in calculating $\int\int_S \mathbf{F}\cdot d\mathbf{S}$, we do not need to calculate $||\mathbf{T}_u\times\mathbf{T}_v||$, even though this quantity is implicit in the definition of $\int\int_S \mathbf{F}\cdot d\mathbf{S}$. Note also, that ultimately, the calculation of $\int\int_S \mathbf{F}\cdot d\mathbf{S}$ reduces to the calculation of a double integral of a function of u and v over the flat region D in the uv-plane. Moreover, $\mathbf{F}(G(u,v))$ is obtained by replacing each occurrence of x, y, z in the definition of $\mathbf{F}$ by $x(u,v), y(u,v), z(u,v)$ respectively.

Example 2. Calculate $\int\int \mathbf{F}\cdot d\mathbf{S}$ for S the upper hemisphere of the sphere of radius R centered at the origin, with the standard parametrization, and $\mathbf{F} = zi + xj + k$.

Solution. We have

$$G(\phi,\theta) = (R\sin(\phi)\cos(\theta), R\sin(\phi)\sin(\theta), R\cos(\phi)),$$

with $0 \le \phi \le \frac{\pi}{2}, 0 \le \theta < 2\pi$, which defines D.

$$\mathbf{T}_\phi = (R\cos(\phi)\cos(\theta), R\cos(\phi)\sin(\theta), -R\sin(\phi)) \quad \text{and} \quad \mathbf{T}_\theta = (-R\sin(\phi)\sin(\theta), R\sin(\phi)\cos(\theta), 0).$$

Thus,

$$\mathbf{T}_\phi \times \mathbf{T}_\theta = \begin{vmatrix} i & j & k \\ R\cos(\phi)\cos(\theta) & R\cos(\phi)\sin(\theta) & -R\sin(\phi) \\ -R\sin(\phi)\sin(\theta) & R\sin(\phi)\cos(\theta) & 0 \end{vmatrix}$$
$$= R^2\sin(\phi)(\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))$$

On the other hand,

$$\mathbf{F}(G(u,v)) = (R\cos(\phi), R\sin(\phi)\cos(\theta), 1),$$

Thus,

$$\begin{aligned}\mathbf{F}(G(u,v)\cdot \mathbf{T}_u \times \mathbf{T}_v &= (R\cos(\phi), R\sin(\phi)\cos(\theta), 1)\cdot R^2\sin(\phi)(\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)) \\ &= R^2\sin(\phi)\{R\cos(\phi)\sin(\phi)\cos(\theta) + R\sin(\phi)\cos(\theta)\sin(\phi)\sin(\theta) + \cos(\phi)\} \\ &= R^3\sin(\phi)\{\cos(\phi)\sin(\phi)\cos(\theta) + \sin^2(\phi)\cos(\theta)\sin(\theta) + \cos(\phi)\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int\int_S \mathbf{F}\cdot d\mathbf{S} &= R^3\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \cos(\phi)\sin^2\phi\cos(\theta) + \sin^3(\phi)\cos(\theta)\sin(\theta) + \cos(\phi)\sin(\phi)\ d\phi\ d\theta \\ &= R^3\int_0^{\frac{\pi}{2}}\int_0^{2\pi} \cos(\phi)\sin^2\phi\cos(\theta) + \sin^3(\phi)\cos(\theta)\sin(\theta) + \cos(\phi)\sin(\phi)\ d\theta\ d\phi \\ &= R^3\int_0^{\frac{\pi}{2}} \cos(\phi)\sin^2(\phi)(-\sin(\theta))_{\theta=0}^{\theta=2\pi} + \sin^3(\phi)(\frac{1}{2}\sin^2(\theta))_{\theta=0}^{\theta=2\pi} + \cos(\phi)\sin(\phi)\theta|_{\theta=0}^{\theta=2\pi}\ d\phi \\ &= 2\pi R^3\int_0^{\frac{\pi}{2}} \cos(\phi)\sin(\phi)\ d\phi \\ &= \pi R^3\sin^2(\phi)\Big|_0^{\phi=\frac{\pi}{2}} \\ &= \pi R^3.\end{aligned}$$

Thursday, April 22. We continued our discussion of line and surface integrals of vector fields by recalling the definitions of each, and the formulas for calculating each, as given in the previous lecture. We emphasized that the definition of the line integral of a vector field along a curve (path) means that we are integrating the tangential component of the vector field along the curve, and that this reduces to just a single integral of a scalar function in one variable. Similarly, the definition of a vector field over a surface means that we integrate the normal (to the surface) components of the vector field over the surface, and hence ultimately, we are calculating a double integral of a scalar function over a subset of $\mathbb{R}^2$. We then worked various examples.

Example 1. $\int_C \mathbf{F}\cdot d\mathbf{r}$ for $\mathbf{F} = x^2i - zj - yz^{-1}k$, for C given by $\mathbf{r}(t) = (t^2, t^3, t)$, with $1 \le t \le 2$.

Solution. $\mathbf{r}'(t) = (2t, 3t^2, 1)$ and $\mathbf{F}(\mathbf{r}(t)) = ((t^2)^2, -t, -t^3t^{-1}) = (t^4, -t, -t^2)$. Thus,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_1^2 \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}(t)\ dt \\ &= \int_1^2 2t^5 - 3t^3 - t^2\ dt \\ &= \{\frac{t^6}{3} - \frac{3t^4}{4} - \frac{t^3}{3}\}_{t=1}^{t=2} \\ &= \frac{89}{12}.\end{aligned}$$

The next example illustrates an interesting phenomenon we will talk about in a few lectures.

Example 2. Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$ for $\mathbf{F} = (3y+1, 3x)$, first for $C_1 : \mathbf{r}(t) = (\cos(t), \sin(t))$ with $0 \le t \le 2\pi$ and then for $C_2 : \mathbf{s}(t) = (1-t, t)$, with $0 \le t \le 1$. Note that both curves begin at the point (1,0) and end at the point (0,1).

Solution. $\mathbf{r}'(t) = (-\sin(t), \cos(t)$ and $\mathbf{F}(\mathbf{r}(t)) = (3\sin(t), 1 + 3\cos(t))$. Thus,

$$\begin{aligned}
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}(t)\ dt \\
&= \int_0^{\frac{\pi}{2}} -3\sin^2(t) - \sin(t) + 3\cos^2(t)\ dt \\
&= \int_0^{\frac{\pi}{2}} 3(\cos^2(t) - \sin^2(t)) - \sin(t)\ dt \\
&= \int_0^{\frac{\pi}{2}3} \cos(2t) - \sin(t)\ dt \\
&= (\frac{3}{2}\sin(2t) + \cos(t))\Big|_0^{\frac{\pi}{2}} \\
&= -1.
\end{aligned}$$

$\mathbf{s}'(t) = (-1, 1)$ and $\mathbf{F}(\mathbf{s}(t)) = (3t + 1, 3(1 - t))$. Thus,

$$\begin{aligned}
\int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}(t)\ dt \\
&= \int_0^1 -(3t + 1) + 3(1 - t)\ dt \\
&= \int_0^1 2 - 6t\ dt \\
&= (2t - 3t^2)\Big|_0^1 \\
&= -1.
\end{aligned}$$

Notice that the two line integrals are the same, even though the paths connecting the endpoints are different. This is a property of the vector field $\mathbf{F}$, called *path independence*. In other words, a vector field $\mathbf{F}$ is path independent if given two points P, Q, $\int_C \mathbf{F} \cdot d\mathbf{r}$ is always the same, no matter which curve we take connecting P and Q. We will talk more about this phenomenon after Exam 3.

In general $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends on the path describing C, even if the parametrization $\mathbf{r}(t)$ is one-to-one. That is because two different parametrizations could yield tangent vectors pointing in opposite directions. For example, we can parametrize the unit circle centered at the origin in $\mathbb{R}^2$ either as $C_1 : \mathbf{r}(t) = (\cos(t), \sin(t))$ or $C_2 : \mathbf{s}(t) = (\cos(2\pi - t), \sin(2\pi - t))$, with $0 \le t \le 2\pi$. We then have that $\mathbf{r}'(t) = (-\sin(t), \cos(t))$ and $\mathbf{s}'(t) = (\cos(2\pi - t), -\sin(2\pi - t))$. Using the subtraction law for sines and cosines, one sees that

$$\mathbf{s}'(t) = (\sin(t), -\cos(t)) = -\mathbf{r}'(t).$$

It follows that for any vector field, $\mathbf{F}$, $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. This property holds in general whenever C_2 traverses the same path as C_1, but in the opposite direction.

We ended class with the following example, given to the class to think about.

Example 3. Let C be the unit circle, traced out once, and $\mathbf{F} = \mathbf{T}$, the unit tangent at each point along the circle. Try to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ without writing anything down, or at least writing very little down.

Solution. 2π. Why: Since $\int_C \mathbf{F} \cdot d\mathbf{r}$ means we integrate the tangential component of $\mathbf{F}$ along the curve C, in this case we are just integrating $\mathbf{T} \cdot \mathbf{T} = 1$ along the curve, which gives the arc length of the curve, i.e., 2π.

Friday, April 23. Continuing our discussion of line and surface integrals of vector fields, we began by reviewing the definition of the surface integral of a vector field and the technique for calculating the surface integral. We also noted that a parametrization of a given surface determines a unit normal vector, and thus $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ can depend upon the paramterization, even if the parametrization $G(u, v)$ is one-to-one. In particular, we noted that $\int\int_{S_2} \mathbf{F} \cdot d\mathbf{S} = -\int\int_{S_1} \mathbf{F} \cdot d\mathbf{S}$, if S_2 is a re-parametrization of S_1 obtained by reversing the unit normal vector. We call S an *oriented surface* if we choose a parametrization that gives a desired system of normal vectors, e.g., outward unit normal vectors to a closed surface like a sphere. For example, if we take the standard parametrization of the sphere of radius R centered at the origin, then the normal vector $\mathbf{T}_u \times \mathbf{T}_v = R^2 \sin(\phi) \cdot (\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))$ is an outward normal, since $(\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))$ is the position vector for points on the unit sphere centered at the origin. We then calculated some examples of surface integrals.

Example 1. Calculate $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ for $\mathbf{F} = 2xi - 2yj + z^2k$, and S the cylinder $0 \le x^2 + y^2 \le 4$, $0 \le z \le 2$, oriented with the outward unit normal.

Solution. We take $G(u, v) = (2\cos(u), 2\sin(u), v)$, with $0 \le u \le 2\pi$ and $0 \le v \le 2$. Then

$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} i & j & k \\ -2\sin(u) & 2\cos(u) & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2\cos(u)i + 2\sin(u)j = (2\cos(u), 2\sin(u), 0).$$

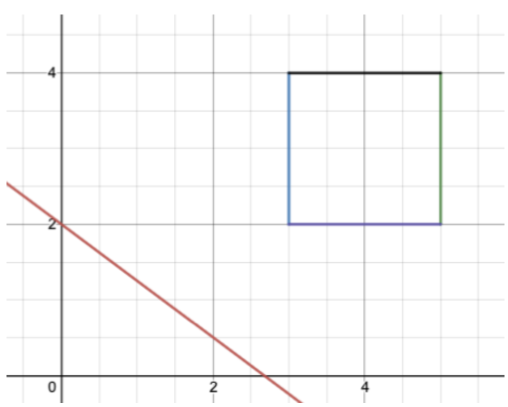
We noted that this gives the outward pointing normal vector, since in the uv-plane, this corresponds to the position vector for the circle of radius two centered at the origin.

We also have $\mathbf{F}(g(u, v)) = (2(2\cos(u), -2(2\sin(u), v^2) = (4\cos(u), -4\sin(u), v^2)$. Thus,

$$\begin{aligned} \int\int_S \mathbf{F} \cdot d\mathbf{S} &= \int\int_D \mathbf{F}(G(u, v)) \cdot \mathbf{T}_u \times \mathbf{T}_v \; dA \\ &= \int_0^2 \int_0^{2\pi} 8\cos^2(u) - 8\sin^2(u) \; du \; dv \\ &= 16 \int_0^{2\pi} \cos(2u) \; du \\ &= 16 \cdot \frac{1}{2} \sin(u) \Big|_0^{2\pi} \\ &= 0. \end{aligned}$$

Example 2. Calculate $\int\int \mathbf{F} \cdot d\mathbf{S}$, with respect to the upward unit normal, for S that portion of the plane $3x + 4y - z = 8$ lying over the rectangle $[3, 5] \times [2, 4]$ and $\mathbf{F} = e^x i + yj + zk$.

Solution. We first note that since the surface S intersects the xy-plane in the line $3x + 4y = 8$, and lies above the xy-plane for points (x, y) lying above this line, the surface S lies entirely over the given rectangle.



We have $G(u, v) = (u, v, 3u + 4v - 8)$, since $z = 3x + 4y - 8$ on the surface.

$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 3 \\ 0 & 1 & 4 \end{vmatrix} = -3i - 4j + k = (-3, -4, 1).$$

Note that since the k component of $\mathbf{T}_u \times \mathbf{T}_v$ is positive, the normal vector $\mathbf{T}_u \times \mathbf{T}_v$ points upward from the surface. We also have $\mathbf{F}(G(u,v)) = (e^u, v, 3u + 4v - 8)$. Thus,

$$\begin{aligned}
\int\int_S \mathbf{F} \cdot d\mathbf{S} &= \int\int_D \mathbf{F}(G(u,v)) \cdot \mathbf{T}_u \times \mathbf{T}_v \; dA \\
&= \int_2^4 \int_3^5 -3e^u - 4v + 3u + 4v - 8 \; du \; dv \\
&= \int_2^4 \int_3^5 3u - 8 - 3e^u \; du \; dv \\
&= 2 \cdot \int_3^5 3u - 8 - 3e^u \; du \\
&= 16 + 6(-e^5 + e^3).
\end{aligned}$$

Example 3. Calculate $\int\int_S \mathbf{F} \cdot d\mathbf{S}$, with respect to the outward normal, for $\mathbf{F} = xi + yj + zk$, and S the closed hemisphere of radius 2 sitting on the xy-plane.

Solution. Note that $S = S_1 \cup S_2$, where: S_1 is the hemisphere and S_2 is the disk $0 \leq x^2 + y^2 \leq 4$, so that $\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int_{S_1} \mathbf{F} \cdot d\mathbf{S} + \int\int_{S_2} \mathbf{F} \cdot d\mathbf{S}$. Now, to calculate $\int\int_{S_1} \mathbf{F} \cdot d\mathbf{S}$, we note that $\mathbf{F} \cdot \mathbf{T}$ over S_1 is just 2, since at each point on the hemisphere, $\mathbf{T}$ is the unit vector pointing in the same direction as $\mathbf{F}$, so that $\mathbf{F} \cdot \mathbf{T} = ||\mathbf{F}|| \cos(\theta) = 2$. Thus,

$$\begin{aligned}
\int\int_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \int\int_{S_1} \mathbf{F} \cdot \mathbf{T} \; dS \\
&= \int\int_{S_1} 2 \; dS \\
&= 2 \cdot \text{surface area}(S_1) \\
&= 16\pi.
\end{aligned}$$

One can check this using the standard parametrization of the sphere of radius 2.

For S_2, the disk in $\mathbb{R}^2$ of radius 2 centered at the origin, we take $\mathbf{n} = -k$ since both $k, -k$ are normal to the disk, but $-k$ points outward from the closed hemisphere. Thus, $\mathbf{F} \cdot \mathbf{n} = -z$. Therefore,

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int_{S_2} z \; dS = 0,$$

since $z = 0$ on the disk.

Alternately, we can use the parametrization $G(u,v) = (u, v, 0)$, with $(u,v) \in D$, where D is the disk of radius 2 in the uv-plane. Therefore, $\mathbf{F}(G(u,v)) = (u, v, 0)$.

$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = k = (0,0,1), \text{ the inward normal vector.}$$

Thus, we must take $-k$ for the outward normal. Therefore,

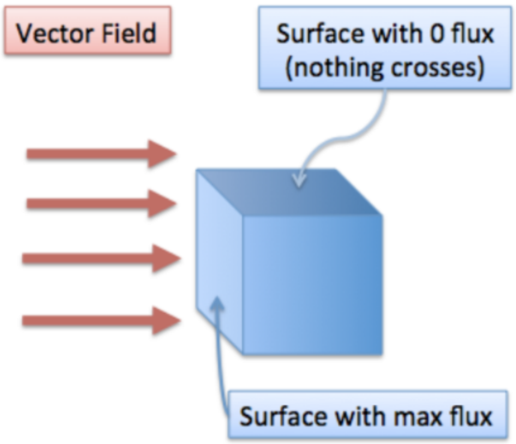
$$\begin{aligned}
\int\int_S \mathbf{F} \cdot d\mathbf{S} &= \int\int_D \mathbf{F}(G(u,v)) \cdot \mathbf{T}_u \times \mathbf{T}_v \; dA \\
&= \int\int_D (u, v, 0) \cdot (0, 0, -1) \; dA \\
&= \int\int_D 0 \; dA \\
&= 0.
\end{aligned}$$

It follows that $\int\int_S \mathbf{F} \cdot d\mathbf{S} = 16\pi + 0 = 16\pi$.

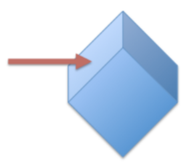
Monday, April 26. We began class with a discussion of the flux of a vector field across a surface. If we imagine that the vector field $\mathbf{F}$ represents a fluid passing through a membrane (e.g., blood through capillaries or an electromagnetic field passing though a shield) then, by definition, the flux is the quantity that passes through the membrane or shield. We want to see that the flux can be calculated as a surface integral.

Case 1. First consider a constant vector field $\mathbf{F}$ passing through a rectangle R perpendicular to the vectors in $\mathbf{F}$. How much of $\mathbf{F}$ passes through R? Answer:

$$||\mathbf{F}|| \cdot \text{area}(R) = (\mathbf{F} \cdot \frac{1}{||\mathbf{F}||}\mathbf{F}) \cdot \text{area}(R) = (\mathbf{F} \cdot \mathbf{n}) \cdot \text{area}(R).$$



Note when $\mathbf{F}$ is parallel to R, the flux is zero. In this case $\mathbf{F} \cdot \mathbf{n} = 0 =$ flux. Case 2. $\mathbf{F}$ is constant, passing through R, neither perpendicular or parallel to R.

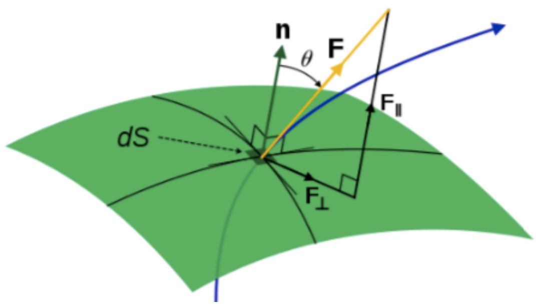


The amount of $\mathbf{F}$ flowing though R is $(\mathbf{F} \cdot \mathbf{n}) \cdot \text{area}(R)$.

General case. $\mathbf{F}$ is not constant, and the surface S is not a rectangle. Subdivide S into very small sections S_i of size dS. For $P_i \in S_i$, $\mathbf{F}$ on S_i is approximately $\mathbf{F}(P_i)$. The flux across each S_i is approximately: $\mathbf{F}(P_i) \cdot \mathbf{n}(P_i)\ dS$. Adding over the surface gives a Riemann sum:

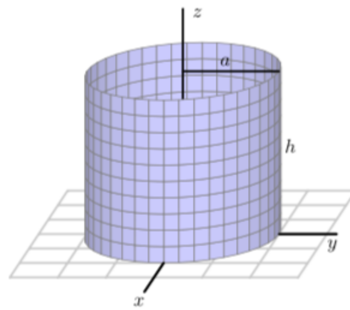
$$\sum_i \mathbf{F}(P_I) \cdot \mathbf{n}(P_i)\ dS.$$

Taking the limit as $dS \to 0$ gives $\int\int_S \mathbf{F} \cdot \mathbf{n}\ dS = \int\int_S \mathbf{F} \cdot d\mathbf{S}$.



$$\text{Flux} = \int\int_S \mathbf{F} \cdot \mathbf{n}\ dS.$$

Example 1. Find the flux of the vector field $\mathbf{F} = y\vec{i} + z\vec{j} + x\vec{k}$, with respect to the outward normal, across the closed cylinder S.



Solution. First, we note that S is the union of three surfaces, the top, bottom and side of the closed cylinder and therefore, the flux of $\mathbf{F}$ across S will be the sum of three surface integrals. Let's minimize the work we have to do. For the top: The unit outward normal is k. Thus $\mathbf{F} \cdot \mathbf{n} = x$ on the top. $\int\int_{\text{Top}} x\ dS = 0$, by symmetry of x with respect to the y-axis for points in a closed disk of radius a. The same reasoning shows $\int\int_{\text{Base}} \mathbf{F} \cdot \mathbf{n}\ dS = 0$, since $-k$ is the unit outward normal.

For the side S_3: $G(u, v) = (a\cos(u), a\sin(u), v)$, $0 \le u \le 2\pi, 0 \le v \le h$.

$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} i & j & k \\ -a\sin(u) & a\cos(u) & 0 \\ 0 & 0 & 1 \end{vmatrix} = (a\cos(u), a\sin(u), 0).$$

$\mathbf{F}(G(u, v)) \cdot \mathbf{T}_u \times \mathbf{T}_v = (a\sin(u), v, a\cos(u)) \cdot (a\cos(u), a\sin(u), 0) = a^2\cos(u)\sin(u) + av\sin(u)$. We therefore have

$$\begin{aligned} \int\int_{S_3} \mathbf{F} \cdot d\mathbf{S} &= \int\int_D \mathbf{F}(G(u, v)) \cdot \mathbf{T}_u \times \mathbf{T}_v\ dA \\ &= \int_0^h \int_0^{2\pi} a^2\cos(u)\sin(u) + av\sin(u)\ dudv \\ &= \int_0^h \left(\frac{a^2}{2}\sin^2(u) + -av\cos(u)\right)\Big|_{u=0}^{u=2\pi}\ dv \\ &= \int_0^h 0\ dv \\ &= 0. \end{aligned}$$

It follows that $\int\int_S \mathbf{F} \cdot d\mathbf{S} = 0 + 0 + 0 = 0$.

We then discussed the *divergence* of a vector field.

Suppose we wish to interpret how the flow of a vector field $\mathbf{F}$ behaves near a point P. For example, a point source of a radiating electromagnetic field or a sink attracting a fluid in motion. Ideally, this should tell us whether or not, on the whole, the field $\mathbf{F}$ is flowing towards or away from the point P, and with what magnitude. A first attempt might be to consider a small closed sphere S_ϵ of radius ϵ about P, and then take the limit of the flux of $\mathbf{F}$ through S_ϵ as $\epsilon \to 0$,

$$\lim_{\epsilon\to 0} \int\int_{S_\epsilon} \mathbf{F}\cdot d\mathbf{S} = \lim_{\epsilon\to 0} \int\int_{S_\epsilon} \mathbf{F}\cdot \mathbf{n}\ dS.$$

The problem with this, is that since the surface area of S_ϵ tends to zero as $\epsilon \to 0$, this limit will always be zero. If we think of S_ϵ as being full of a liquid, and we empty this liquid as $\mathbf{F}$ moves from P, this all passes through the boundary of S_ϵ as $\epsilon \to 0$. In other words, what we want is the flux per unit volume of $\mathbf{F}$ over S_ϵ, as $\epsilon \to 0$, which would be

$$\lim_{\epsilon\to 0} \frac{1}{\text{vol}(S_\epsilon)} \int\int_{S} \mathbf{F}\cdot d\mathbf{S}.$$

This limit is called the *Divergence* of $\mathbf{F}$ at the point P. Letting P vary, we have a new **scalar** function, called the divergence of $\mathbf{F}$, which we denote by div $\mathbf{F}$.

Important facts about the Divergence of $\mathbf{F}$.

(i) The divergence does not depend upon the coordinate system used.
(ii) The divergence of $\mathbf{F}$ at P can be calculated as

$$\lim_{\text{vol}(S)\to 0} \frac{1}{\text{vol}(S)} \int\int_{S} \mathbf{F}\cdot d\mathbf{S}$$

where the limit is taken over any sequence of (sufficiently smooth) closed surfaces containing P, whose volumes tend to 0.
(iii) div $\mathbf{F}$ is a scalar function of x, y, z.
(iv) $\text{div}(\mathbf{F}+\mathbf{G}) = \text{div }\mathbf{F} + \text{div }\mathbf{G}$.
(v) $\lambda\cdot \text{div }\mathbf{F} = \text{div}(\lambda\mathbf{F})$, for any $\lambda \in \mathbb{R}$.

Example 2. Let $\mathbf{F} = x^3\vec{i} + 4y\vec{j} + 2z^2\vec{k}$. Calculate div $\mathbf{F}(x_0, y_0, z_0)$, using cubes centered at $P = (x_0, y_0, z_0)$.

Solution. Let C_s denote the cube centered at P having sides of length s. Then,

$$\text{div }\mathbf{F}(P) = \lim_{s\to 0} \frac{1}{s^3} \int\int_{C_s} \mathbf{F}\cdot d\mathbf{S} = \lim_{s\to 0} \frac{1}{s^3} \int\int_{C_s} \mathbf{F}\cdot \mathbf{n}\ dS.$$

We first evaluate the surface integral of $\mathbf{F}$ over the front and back faces of C_s. For the front face S_1, we note that $\mathbf{n} = \vec{i}$ and that the x coordinate of every point on the front fact is $x_0 + \frac{s}{2}$. Since $\mathbf{F}\cdot\mathbf{n} = x^3$, it follows that on the from fact, $\mathbf{F}\cdot\mathbf{n} = (x_0 + \frac{s}{2})^3$. Noting that x_0 and s are constants (relative to S_1), it follows that

$$\int\int_{S_1} \mathbf{F}\cdot\mathbf{n}dS = \int\int_{S_1} (x_0 + \frac{s}{2})^3\ dS = (x_0 + \frac{s}{2})^3 \cdot \text{vol}(S_1) = (x_0 + \frac{s}{2})^3 \cdot s^2.$$

The calculation for S_2 is similar. In this case, $\mathbf{n} = -\vec{i}$, $\mathbf{F}\cdot\mathbf{n} = -x^3$ and $-x^3$ evaluate on S_2 is the constant $-(x_0 - \frac{s}{2})^3$. Thus,

$$\int\int_{S_2} \mathbf{F}\cdot\mathbf{n}dS = \int\int_{S_2} -(x_0 - \frac{s}{2})^3\ dS = -(x_0 - \frac{s}{2})^3 \cdot \text{vol}(S_2) = -(x_0 - \frac{s}{2})^3 \cdot s^2.$$

Therefore,

$$\int\int_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS + \int\int_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS = (x_0+\frac{s}{2})^3\cdot s^3 + (x_0-\frac{s}{2})^3\cdot s^2$$
$$= s^2\{(x_0^3+\frac{3}{2}x_0^2 s+\frac{3}{4}x_0 s^2+\frac{s^3}{8})-(x_0^3-\frac{3}{2}x_0^2 s+\frac{3}{4}x_0 s^2-\frac{s^3}{8})\}$$
$$= s^2\{3x_0^2 s+\frac{s^3}{4}\}$$
$$= 3x_0 s^3+\frac{s^5}{4}.$$

We proceed in similar fashion for the next two faces, the right and left faces of C_s. Suppose S_3 is the right face of the cube C_s. Then $\mathbf{n}=\vec{j}$, $\mathbf{F}\cdot\mathbf{n}=4y$, and $4y$ evaluated on S_3 equals $4(y_0+\frac{s}{2})$, a constant on S_3. Thus,

$$\int\int_{S_3} \mathbf{F}\cdot\mathbf{n}\,dS = 4(y_0+\frac{s}{2})\cdot \mathrm{vol}(S_3) = 4(y_0+\frac{s}{2})\cdot s^2.$$

For the left face S_4, $\mathbf{n}=-\vec{j}$, $\mathbf{F}\cdot\mathbf{n}=-4y$ and $-4y$ evaluated on S_3 equals $-4(y_0-\frac{s}{2})$, a constant on S_3. Thus,

$$\int\int_{S_4} \mathbf{F}\cdot\mathbf{n}\,dS = -4(y_0-\frac{s}{2})\cdot \mathrm{vol}(S_3) = -4(y_0-\frac{s}{2})\cdot s^2.$$

Thus,

$$\int\int_{S_3} \mathbf{F}\cdot\mathbf{n}\,dS + \int\int_{S_4} \mathbf{F}\cdot\mathbf{n}\,dS = 4(y_0+\frac{s}{2})\cdot s^2 - 4(y_0-\frac{s}{2})\cdot s^2$$
$$= 4s^3.$$

Finally, for the top face S_5, $\mathbf{n}=\vec{k}$, $\mathbf{F}\cdot\mathbf{n}=2z^2$ and $2z^2$ on S_5 equals $2(z_0+\frac{s}{2})^2$. Thus, as before,

$$\int\int_{S_5} \mathbf{F}\cdot\mathbf{n}\,dS = 2(z_0+\frac{s}{2})^2\cdot \mathrm{vol}(S_5) = 2(z_0+\frac{s}{2})^2\cdot s^2.$$

Similarly, if S_6 denotes the bottom face of C_s, $\mathbf{n}=-\vec{k}$, $\mathbf{F}\cdot\mathbf{n}=2z^2$, which equals $2(z_0-\frac{s}{2})^2$ on S_6. Thus,

$$\int\int_{S_5} \mathbf{F}\cdot\mathbf{n}\,dS = -2(z_0-\frac{s}{2})^2\cdot \mathrm{vol}(S_5) = -2(z_0-\frac{s}{2})^2\cdot s^2.$$

Therefore,

$$\int\int_{S_5} \mathbf{F}\cdot\mathbf{n}\,dS + \int\int_{S_6} \mathbf{F}\cdot\mathbf{n}\,dS = 2(z_0+\frac{s}{2})^2\cdot s^2 - 2(z_0-\frac{s}{2})^2\cdot s^2$$
$$= 4z_0 s^3.$$

Adding these six integrals, we now have

$$\int\int_{C_s} \mathbf{F}\cdot\mathbf{n}\,dS = 3x_0 s^3+\frac{s^5}{4}+4s^3+4z_0 s^3$$

Therefore,

$$\mathrm{div}\ \mathbf{F}(P) = \lim_{\mathrm{vol}(C_s)\to 0} \frac{1}{\mathrm{vol}(C_s)} \int\int_{C_s} \mathbf{F}\cdot\mathbf{n}\,dS$$
$$= \lim_{s\to 0} \frac{1}{s^3}\cdot(3x_0 s^3+\frac{s^5}{4}+4s^3+4z_0 s^3)$$
$$= 3x_0+4+4z_0.$$

As one might guess from the preceding example, here is the formula for the divergence in rectangular coordinate:

$$\mathrm{div}\ \mathbf{F}(P) = \frac{\partial F_1}{\partial x}(P)+\frac{\partial F_2}{\partial y}(P)+\frac{\partial F_3}{\partial z}(P).$$

Example 3. Calculate div $\mathbf{F}$ for $\mathbf{F} = 3x^3yz\vec{i} + e^{x+2y+3z^2}\vec{j} + z\cos(xy)\vec{k}$.

Solution. div $\mathbf{F} = 9x^2yz + 2e^{x+2y+3z^2} + \cos(xy)$.

In rectangular coordinates, we often write $\nabla \cdot \mathbf{F}$ for div $\mathbf{F}$, since we think of ∇ as $\frac{\partial}{\partial x}\ \vec{i} + \frac{\partial}{\partial y}\ \vec{j} + \frac{\partial}{\partial z}\ \vec{k}$, which is a *differential operator*. Thus,

$$\begin{aligned}\nabla \cdot \mathbf{F} &= (\frac{\partial}{\partial x}\ \vec{i} + \frac{\partial}{\partial y}\ \vec{j} + \frac{\partial}{\partial z}\ \vec{k}) \cdot (F_1\vec{i} + F_2\vec{j} + F_3\vec{k}) \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\ &= \text{div } \mathbf{F}.\end{aligned}$$

The use of ∇ in this way is consistent with our prior use, if we recall that the gradient of a scalar function $f(x, y, z)$ is:

$$\nabla(f) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}.$$

The formula for div $\mathbf{F}$ takes different forms in different coordinate systems.

In cylindrical coordinates: $\mathbf{F} = (F_r, F_\theta, F_z)$ (**not** $F_r i + F_\theta j + F_z k$), and

$$\text{div } \mathbf{F} = \frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}.$$

In spherical coordinates: $\mathbf{F} = (F_r, F_\phi, F_\theta)$, and

$$\text{div } \mathbf{F} = \frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r} + \frac{1}{r\sin(\theta)}\frac{\partial F_\phi}{\partial \phi} + \frac{1}{\sin(\theta)}\frac{\partial(\sin(\theta)F_\theta)}{\partial \theta}.$$

Tuesday, April 27. We started class with Quiz 10. We then reviewed the definition of the divergence of a vector field, and began our discussion of the Divergence theorem.

Given a vector field $\mathbf{F}$, the divergence of $\mathbf{F}$ at a point P measures the flux per unit volume at the point P. The divergence of $\mathbf{F}$ at P is defined by the equation:

$$\text{div } \mathbf{F}(P) = \lim_{\text{vol}(S)\to 0} \frac{1}{\text{vol}(S)} \int\int_S \mathbf{F} \cdot d\mathbf{S}$$

where the limit is taken over any sequence of closed (sufficiently smooth) surfaces containing P whose volumes tend to 0.

How do we calculate div $\mathbf{F}$?

Answer:The calculation depends upon the coordinate system. In rectangular coordinates,

$$\text{div } \mathbf{F}(P) = \frac{\partial F_1}{\partial x}(P) + \frac{\partial F_2}{\partial y}(P) + \frac{\partial F_3}{\partial z}(P).$$

Here is one of the main theorems of our course.

The Divergence Theorem. Let S be a closed (piece-wise smooth) surface that bounds the solid B in $\mathbb{R}^3$. If the first order partial derivatives of the component functions of $\mathbf{F}$ are continuous on B, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int\int_B \text{div } \mathbf{F}\ dV.$$

Note that the integral on the right is a standard triple integral of a scalar function over a solid in $\mathbb{R}^3$, while the integral on the left is a surface integral of the the vector field $\mathbf{F}(x, y, z)$ with respect to the outward pointing normal vectors.

Example 1. For $\mathbf{F} = x\vec{i} + y\vec{j} + z\vec{k}$ and S is the sphere of radius R centered at the origin, calculate both terms in the Divergence Theorem.

Solution. For the sphere, with the usual parametrization $G(\phi, \theta)$, $\mathbf{F}(G(\phi, \theta)) = (R\sin(\phi)\cos(\theta), R\sin(\phi)\sin(\theta), R\cos(\phi))$. And, $\mathbf{T}_\phi \times \mathbf{T}_\theta = R^2\sin(\phi)(\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))$, the outward normal vector.

$\mathbf{F}(G(\phi, \theta)) \cdot \mathbf{T}_\phi \times \mathbf{T}_\theta = R^3 \sin\phi$. Therefore,

$$\begin{aligned}\int\int_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{\pi} \mathbf{F}(G(\phi,\theta)) \cdot \mathbf{T}_\phi \times \mathbf{T}_\theta \; d\phi d\theta \\ &= \int_0^{2\pi}\int_0^{\pi} R^3 \sin(\phi) \; d\phi d\theta \\ &= 2\pi R^3 \int_0^{\pi} \sin(\phi) \; d\phi \\ &= 4\pi R^3.\end{aligned}$$

Alternately, the unit normal vector $\mathbf{n}$ on S is $\frac{x}{R}\vec{i} + \frac{y}{R}\vec{j} + \frac{z}{R} \; vk$. Thus, $\mathbf{F} \cdot \mathbf{n} = \frac{x^2}{R} + \frac{y^2}{R} + \frac{z^2}{R}$, which equals R when evaluate on S. Thus,

$$\begin{aligned}\int\int_S \mathbf{F} \cdot \mathbf{n} \; dS &= \int\int_S R \; dS \\ &= R \cdot (\text{surface area}(S)) \\ &= 4\pi R^3.\end{aligned}$$

On the other hand: div $\mathbf{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$.

$$\begin{aligned}\int\int\int_B \text{div } \mathbf{F} \; dV &= \int\int\int_B 3 \; dV \\ &= 3 \cdot \text{vol}(W) \\ &= 3 \cdot \frac{4}{3}\pi R^3 \\ &= 4\pi R^3.\end{aligned}$$

Thus, $\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int\int_B \text{div } \mathbf{F} \; dV$.

Example 2. Calculate $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ with respect to the outward normal vectors, for $\mathbf{F} = xz^2 i + yx^2 j + zy^2 k$ and S the sphere of radius R centered at the origin.

Solution. We can set up the surface integral by regarding it as $\int\int_S \mathbf{F} \cdot \mathbf{n} \; dS$. For a point $P = (x, y, z)$ on S, the unit normal at P is $\frac{1}{R}(x, y, z)$. Thus $\mathbf{F} \cdot \mathbf{n}$ on S equals

$$(xz^2, yx^2, zy^2) \cdot \frac{1}{R}(x, y, z) = \frac{x^2z^2 + y^2x^2 + z^2y^2}{R}.$$

Therefore,

$$\begin{aligned}\int\int_S \mathbf{F} \cdot d\mathbf{S} &= \int\int_S \mathbf{F} \cdot \mathbf{n} \; dS \\ &= \int\int_S \frac{x^2z^2 + y^2x^2 + z^2y^2}{R} \; dS,\end{aligned}$$

Which is doable, but not pleasant. Better: Use the Divergence Theorem. div $\mathbf{F} = z^2 + +x^2 + y^2$. For B the solid sphere,

$$\int\int\int_B \text{div } \mathbf{F} \; dV = \int\int\int_B x^2 + y^2 + z^2 \; dV,$$

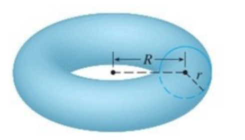
which can easily be calculated using spherical coordinates.

$$\begin{aligned}\int\int\int_B \text{div } \mathbf{F}\ dV &= \int\int\int_B x^2+y^2+z^2\ dV \\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2\cdot\rho^2\sin(\phi)\ d\rho d\phi d\theta \\ &= 2\pi\int_0^{\pi}\int_0^{R}\rho^4\sin(\phi)\ d\rho d\phi \\ &= \frac{2}{5}\pi R^5\int_0^{\pi}\sin(\phi)\ d\phi \\ &= \frac{4}{5}\pi R^5.\end{aligned}$$

Therefore, $\int\int_S \mathbf{F}\cdot d\mathbf{S} = \frac{4}{5}\pi R^5$.

Wednesday, April 28. We continued our discussion of the Divergence Theorem. The first example is a typical application.

Example 1. Calculate $\int\int_S \mathbf{F}\cdot d\mathbf{S}$ with respect to the outward normal, for $\mathbf{F} = yz^3 i + e^{x^2+z^2} j + \cos(\sqrt{x^2+y^2})k$ and S torus obtained by revolving the circle $(y-3)^2+z^2=4$ in the yz-plane about the y-axis,



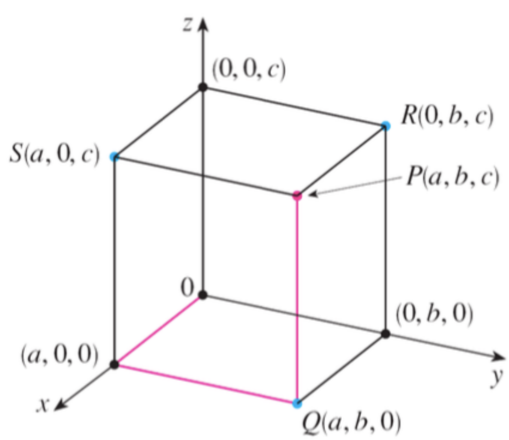
A parametrization for S is: $x=(3+2\cos(v))\sin(u)$, $y=(3+2\cos(v))\cos(u)$, $z=2\sin(v)$, with $0\le u,v\le 2\pi$.

Solution. No worries: div $\mathbf{F}=0$, so

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} = \int\int\int_B \text{div } \mathbf{F}\ dV = \int\int\int_B 0\ dV = 0,$$

where B is the solid enclosed by S.

Example 2. Verify the Divergence Theorem for $\mathbf{F} = x^2 i + y^2 j + z^2 k$, and B the solid rectangle $0\le x\le a$, $0\le y\le b$, $0\le z\le c$.



Solution. We first calculate $\int\int_{S_i} \mathbf{F}\cdot\mathbf{n}\ dS$, for each of the six faces $S_1,\dots,S_6$ forming the boundary of B. For the front face S_1, we have $\mathbf{n}=\vec{i}$, so that $\mathbf{F}\cdot\mathbf{n}=x^2$. Thus $\int\int_{S_1}\mathbf{F}\cdot\mathbf{n}\ dS = \int\int_{S_1} x^2\ dS$. But on S_1,

$x = a$, therefore,

$$\begin{aligned}\int\int_{S_1} \mathbf{F}\cdot\mathbf{n}\ dS &= \int\int_{S_1} x^2\ dS\\ &= \int\int_{S_1} a^2\ dS\\ &= a^2\cdot \text{surface area}(S_1)\\ &= a^2bc.\end{aligned}$$

For the back face S_2, $\mathbf{n} = -\vec{i}$, so $\mathbf{F}\cdot\mathbf{n} = -x^2$. But $x = 0$ on S_2, so that $\int\int_{S_2} \mathbf{F}\cdot\mathbf{n}\ dS = \int\int_{S_2} 0^2\ dS = 0$.

For the right face S_3, $\mathbf{n} = \vec{j}$, so $\mathbf{F}\cdot\mathbf{n} = y^2$. However, $y = b$ on the right face, so that $\mathbf{F}\cdot\mathbf{n} = b^2$ on S_2. Therefore,

$$\begin{aligned}\int\int_{S_3} \mathbf{F}\cdot\mathbf{n}\ dS &= \int\int_{S_3} y^2\ dS\\ &= \int\int_{S_3} b^2\ dS\\ &= b^2\cdot \text{surface area}(S_3)\\ &= ab^2c.\end{aligned}$$

For the left face S_4, $\mathbf{n} = -\vec{j}$, so $\mathbf{F}\cdot\mathbf{n} = -y^2$. But $y = 0$ on S_4, so that $\int\int_{S_4} \mathbf{F}\cdot\mathbf{n}\ dS = \int\int_{S_4} 0^2\ dS = 0$.

For the top face S_5, $\mathbf{n} = \vec{k}$, so $\mathbf{F}\cdot\mathbf{n} = z^2$. However, $z = c$ on the top face, so that $\mathbf{F}\cdot\mathbf{n} = c^2$ on S_2. Therefore,

$$\begin{aligned}\int\int_{S_5} \mathbf{F}\cdot\mathbf{n}\ dS &= \int\int_{S_3} z^2\ dS\\ &= \int\int_{S_5} c^2\ dS\\ &= c^2\cdot \text{surface area}(S_5)\\ &= abc^2.\end{aligned}$$

For the bottom face S_6, $\mathbf{n} = -\vec{k}$, so $\mathbf{F}\cdot\mathbf{n} = -z^2$. But $z = 0$ on S_6, so that $\int\int_{S_6} \mathbf{F}\cdot\mathbf{n}\ dS = \int\int_{S_6} 0^2\ dS = 0$.

Putting this all together, we have:

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} = a^2bc + 0 + ab^2c + 0 + abc^2 + 0 = abc(a+b+c).$$

We now have div $\mathbf{F} = 2x + 2y + 2z$. Therefore,

$$\begin{aligned}\int\int\int_B \text{div } \mathbf{F}\ dV &= \int\int\int_B 2x + 2y + 2x\ dV\\ &= \int_0^c\int_0^b\int_0^a 2x + 2y + 2z\ dV\\ &= \int_0^c\int_0^b a^2 + 2ya + 2za\ dV\\ &= \int_0^c a^2b + b^2a + 2zab\ dV\\ &= a^2bc + b^2ac + c^2ab\\ &= abc(a+b+c),\end{aligned}$$

which is what we want.

Why does the Divergence Theorem work? We want to compare $\int\int\int_B \text{div } \mathbf{F}\ dV$ and $\int\int_S \mathbf{F}\cdot d\mathbf{S}$.

Step 1. Working in rectangular coordinates, we subdivide B into small solids solids B_i that are approximately cubes, each with volume ΔV.

Step 2. We take a point P_i in each B_i.

Step 3. div $\mathbf{F}(P_i) \approx \frac{1}{\text{vol}(B_i)} \cdot \int\int_{S_i} \mathbf{F}\cdot\mathbf{n}\, dS$, where S_i is the boundary of B_i.

Step 4. div $\mathbf{F}(P_i)\cdot \text{vol}(B_i) \approx \int\int_{S_i} \mathbf{F}\cdot\mathbf{n}\, dS$.

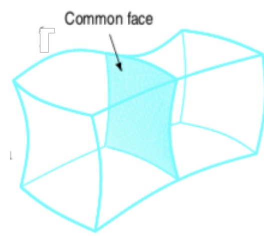
Summing the left hand side we get:

$$\sum_i \text{div } \mathbf{F}(P_i)\cdot \text{vol}(B_i) = \sum_i \text{div } \mathbf{F}(P_i)\Delta V,$$

a Reimann sum. Passing to the limit as $\Delta V \to 0$ we get:

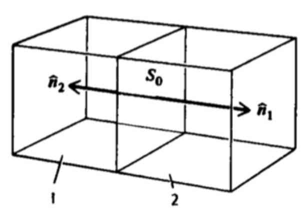
$$\int\int\int_B \text{div } \mathbf{F}\, dV.$$

Something interesting happens when we sum the terms belonging to the right hand side of Step 4.



When we integrate $\mathbf{F}$ over the boundaries of adjacent V_i, the surface integrals over common faces cancel.

Why: We integrate twice over the common face, once for S_1 and again for S_2, but once with a normal vector $\mathbf{n}$ and again with $-\mathbf{n}$. For the common face S_0:



$\int\int_{S_0} \mathbf{F}\cdot\mathbf{n_1}\, dS = \int\int_{S_0} \mathbf{F}\cdot -\mathbf{n_2}\, dS$. Thus when we add

$$\int\int_{S_1} \mathbf{F}\cdot\mathbf{n}\, dS + \int\int_{S_2} \mathbf{F}\cdot\mathbf{n}\, dS,$$

the components over S_0 cancel and the sum becomes $\int\int_{\tilde{S}} \mathbf{F}\cdot\mathbf{n}\, dS$, where $\tilde{S}$ is the combined outer shell of S_1 and S_2.

Thus, the sum of the terms in the RHS in Step 4 approximates $\int\int_S \mathbf{F}\cdot d\mathbf{S}$, and equals it in the limit.

Here is another application of the Divergence Theorem.

Gauss' Law (Physics Version). The net electric flux through any hypothetical closed surface is equal to $\frac{1}{\epsilon 0}$ times the net electric charge within that closed surface, where ϵ_0 is the electric constant.

Gauss' Law (Math Version). Let M be a solid in $\mathbb{R}^3$ with a smooth boundary ∂M. Assume (0,0,0) is not on the boundary ∂M. Set $\mathbf{r} = xi + yj + zk$, and $r = ||\mathbf{r}|| = \sqrt{x^2+y^2+z^2}$. Then:

$$\int\int_{\partial M} (\frac{1}{r^3}\mathbf{r})\cdot\mathbf{n}\, dS = \begin{cases} 4\pi, & if\ (0,0,0) \in M \\ 0, & if\ (0,0,0) \notin M \end{cases}.$$

In the second case, the vector field $\frac{1}{r^3}\mathbf{r}$ is defined throughout M, so that if we show its divergence is zero, then by the Divergence Theorem,

$$\int\int_{\partial M}(\frac{1}{r^3}\mathbf{r})\cdot\mathbf{n}\,dS = 0.$$

The i component of $\frac{1}{r^3}\mathbf{r}$ is $\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}$. Differentiating with respect to x we get:

$$\frac{(x^2+y^2+z^2)^{\frac{3}{2}} - x\cdot\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}(2x)}{(x^2+y^2+z^2)^3} =$$

$$(x^2+y^2+z^2)^{\frac{1}{2}}\cdot\frac{(x^2+y^2+z^2)-3x^2}{(x^2+y^2+z^2)^3} =$$

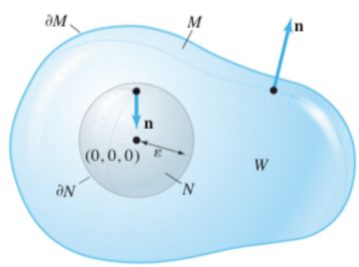
$$\frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}.$$

Similarly, the corresponding partials of the j and k components are:

$$\frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} \quad\text{and}\quad \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}.$$

Adding these three terms shows $\text{div}(\frac{1}{r^3}\mathbf{r}) = 0$.

Now, suppose $(0,0,0)\in M$. Let N be a sphere of radius ϵ centered at (0,0,0) contained in M. Let W be the complement of N in M. The boundary of W is $\partial N\cup\partial M$, i.e., the union of the boundaries of M and N.



Since $\text{div}\frac{1}{r^3}\mathbf{r} = 0$ over W, $\int\int_{\partial N\cup\partial M}\frac{1}{r^3}\mathbf{r}\cdot\mathbf{n}\,dS = 0$. Therefore,

$$\int\int_{\partial M}\frac{1}{r^3}\mathbf{r}\cdot\mathbf{n}\,dS = -\int\int_{\partial N}\frac{1}{r^3}\mathbf{r}\cdot\mathbf{n}\,dS.$$

Note that the outward normal for ∂N as part of the boundary of B is the inward normal for N. Thus,

$$\int\int_{\partial M}\frac{1}{r^3}\mathbf{r}\cdot\mathbf{n}\,dS = \int\int_{\partial N}\frac{1}{r^3}\mathbf{r}\cdot\mathbf{n}\,dS,$$

where now $\mathbf{n}$ is the outward normal for the sphere N.

On the sphere ∂N, the sphere of radius ϵ centered at the origin, $\mathbf{n} = \frac{1}{\epsilon}\mathbf{r}$ and $r = \epsilon$. Then:

$$\frac{1}{r^3}\mathbf{r}\cdot\mathbf{n} = \frac{1}{\epsilon^3}\mathbf{r}\cdot\frac{1}{\epsilon}\mathbf{r} = \frac{\epsilon^2}{\epsilon^4} = \frac{1}{\epsilon^2}.$$

Thus:

$$\begin{aligned}\int\int_{\partial M}\frac{1}{r^3}\mathbf{r}\cdot\mathbf{n}\,dS &= \int\int_{\partial N}\frac{1}{r^3}\mathbf{r}\cdot\mathbf{n}\,dS\\ &= \int\int_{\partial N}\frac{1}{\epsilon^2}\,dS\\ &= \frac{1}{\epsilon^2}\cdot\text{area}(\partial N) = \frac{1}{\epsilon^2}\cdot 4\pi\epsilon^2\\ &= 4\pi.\end{aligned}$$

Final Comment: If we use the symbol ∂B to denote the boundary of the solid in the Divergence Theorem, and the notation $\nabla \cdot \mathbf{F}$ for div $\mathbf{F}$, then the Divergence Theorem becomes:

$$\int\int_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \int\int\int_{B} \nabla \cdot \mathbf{F}\ dV.$$

Note how the differential operator on the domain of integration on the left hand side of the equation, moves up to the integrand on the right hand side of the equation.

Thursday, April 29. We worked in Breakout Rooms on the practice problems for Exam 3.

Friday, April 20. We continued our review for Exam 3 by working in Breakout Rooms on the practice problems for Exam 3. Solutions to the practice problems were posted later in the evening.

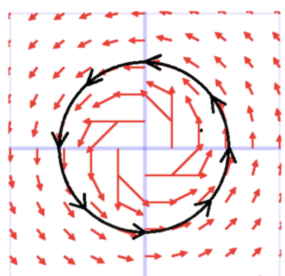
Monday, May 3. Exam 3.

Tuesday, May 4. We discussed the curl of a vector field in $\mathbb{R}^2$ and Green's Theorem.

We start with a vector field $\mathbf{F} = F_1(x,y)\vec{i} + F_2(x,y)\vec{j}$ in $\mathbb{R}^2$. If C is a closed path, the circulation of $\mathbf{F}$ along C is the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{t}\ ds,$$

where we assume C is oriented in a counter-clockwise direction.



Throughout we assume that the first order partial derivatives of the component functions of $\mathbf{F}$ exist and are continuous. Later we will need to assume that all second order partial derivatives of the components functions exist and are continuous.

Definition. The curl of $\mathbf{F}$ at the point P is the circulation per unit area at the point P and is given by the formula

$$(\text{Curl } \mathbf{F})(P) = \lim_{\text{area}(D) \to 0} \frac{1}{\text{area}(D)} \int_{\partial D} \mathbf{F} \cdot d\mathbf{r},$$

where D is a region in $\mathbb{R}^2$ whose boundary ∂D is a simple closed curve. A simple closed curve is a (piecewise) smooth curve with no self intersections. We have the following properties of the curl of $\mathbf{F}$:

(i) Curl $\mathbf{F}$ is a scalar function.
(ii) We can calculate (Curl $\mathbf{F}$)(P) using shrinking regions of our choice.
(iii) The value of (Curl $\mathbf{F}$)(P) does not depend upon the regions we choose.
(iv) (Curl $\mathbf{F}$)(P) is also independent of the coordinate system.

Example 1. Calculate (Curl $\mathbf{F}$)(P) for $\mathbf{F} = -yi + xj$ and $P = (x_0, y_0)$, using shrinking disks of radius epsilon.

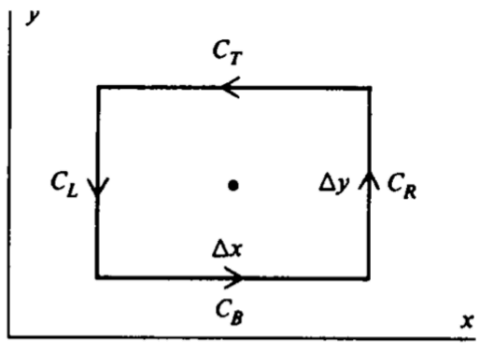
Solution. If we let D_ϵ denote the disk of radius ϵ centered at P, and C_ϵ denote its boundary, then using the parametrization of C_ϵ : $\mathbf{r}(t) = (\epsilon\cos(t) + x_0, \epsilon\sin(t) + y_0)$, with $0 \le t \le \pi$, we have

$$\begin{aligned}
\text{Curl }\mathbf{F} &= \lim_{\epsilon\to 0}\frac{1}{\pi\epsilon^2}\int_{C_\epsilon}\mathbf{F}\cdot\ d\mathbf{r}\\
&= \lim_{\epsilon\to 0}\frac{1}{\pi\epsilon^2}\int_0^{2\pi}(-\epsilon\sin(t) - y_0, \epsilon\cos(t) + x_0)\cdot(-\epsilon\sin(t), \epsilon\cos(t))\ dt\\
&= \lim_{\epsilon\to 0}\frac{1}{\pi\epsilon^2}\int_0^{2\pi}\epsilon^2 + \epsilon y_0\sin(t) + \epsilon x_0\cos(t)\ dt\\
&= \lim_{\epsilon\to 0}\frac{1}{\pi\epsilon^2}\{\epsilon^2 t - \epsilon y_0\cos(t) + \epsilon x_0\sin(t)\}\Big|_0^{2\pi}\\
&= \lim_{\epsilon\to 0}\frac{1}{\pi\epsilon^2}\cdot 2\pi\epsilon^2\\
&= 2.
\end{aligned}$$

How do we calculate the curl using rectangular coordinates? Answer:

$$\text{Curl }\mathbf{F}(P) = \frac{\partial F_2}{\partial x}(P) - \frac{\partial F_1}{\partial y}(P).$$

Why (heuristically): We start with a $(\Delta x)\times(\Delta y)$ square centered at (x_0, y_0). We travel counter-clockwise around the square.



Along the base C_B, the unit tangent is $\vec{i}$, so that $\mathbf{F}\cdot\mathbf{t}$ is F_1 along C_B. Since $y = y_0 - \frac{\Delta y}{2}$ along the base, $\mathbf{F}\cdot\mathbf{t} = F_1(x, y_0 - \frac{\Delta y}{2})$. Moreover, if Δx is small, and x_0 is the center of C_B, then the average value of F_1 along C_B is approximately $F_1(x_0, y_0 - \frac{\Delta y}{2})$. Thus,

$$\int_{C_B}\mathbf{F}\cdot\mathbf{t}\ ds = \int_{C_B}F_1\ ds \approx F_1(x_0, y_0 - \frac{\Delta y}{2})\Delta x.\ (\text{Avg value times length})$$

Along the top C_T, the unit tangent is $-\vec{i}$, so that $\mathbf{F}\cdot\mathbf{t}$ is -F_1 along C_T. Thus, similarly, we have

$$\int_{C_T}\mathbf{F}\cdot\mathbf{t}\ ds = \int_{C_B}-F_1\ ds \approx -F_1(x_0, y_0 + \frac{\Delta y}{2})\Delta x.\ (\text{Avg value times length})$$

Adding these terms we have

$$\int_{C_B+C_T}\mathbf{F}\cdot\mathbf{t}\ ds \approx -\Delta x\cdot\{F_1(x_0, y_0 + \frac{\Delta y}{2}) - F_1(x_0, y - \frac{\Delta y}{2})$$

Dividing by the area $\Delta x\Delta y$, we get:

$$\frac{1}{\Delta x\Delta y}\int_{C_B+C_T}\mathbf{F}\cdot\mathbf{t}\ ds = -\frac{1}{\Delta y}\cdot\{F_1(x_0, y_0 + \frac{\Delta y}{2}) - F_1(x_0, y - \frac{\Delta y}{2})\}.$$

Taking the limit as $\Delta x, \Delta y \to 0$ gives

$$\lim_{\Delta, x, \Delta y \to 0} \frac{1}{\Delta x \Delta y} \int_{C_B + C_T} \mathbf{F} \cdot \mathbf{t} \; ds = -\frac{\partial F_1}{\partial y}(x_0, y_0).$$

A similar analysis yields

$$\lim_{\Delta, x, \Delta y \to 0} \frac{1}{\Delta x \Delta y} \int_{C_R + C_L} \mathbf{F} \cdot \mathbf{t} \; ds = \frac{\partial F_2}{\partial x}(x_0, y_0).$$

Adding we get

$$\begin{aligned}(\text{Curl } \mathbf{F})(x_0, y_0) &= \lim_{\Delta, x, \Delta y \to 0} \frac{1}{\Delta x \Delta y} \int_C \mathbf{F} \cdot \mathbf{t} \; ds \\ &= \frac{\partial F_2}{\partial x}(x_0, y_0) - \frac{\partial F_1}{\partial y}(x_0, y_0).\end{aligned}$$

Example 2. Let us check the formula with Example 1.$\mathbf{F} = -y\vec{i} + xj$. $\frac{\partial F_2}{\partial x} = \frac{\partial x}{\partial x} = 1$. $\frac{\partial F_1}{\partial y} = \frac{\partial -y}{\partial y} = -1$.

Solution.

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1 - (-1) = 2 = \text{Curl } \mathbf{F}.$$

Example 3. Find the curl of $\mathbf{F} = ye^{x-1}i + y^2x^3j$ at (2,1).

Solution.

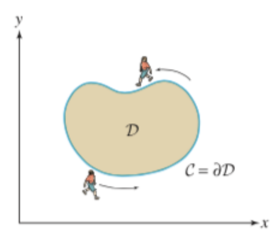
$$\frac{\partial(y^2x^3)}{\partial x} = 3y^2x^2 \quad \text{and} \quad \frac{\partial(ye^{x-1})}{\partial y} = e^{x-1}.$$

Curl $\mathbf{F} = 3y^2x^2 - e^{x-1}$. In particular, $(\text{Curl } \mathbf{F})(2,1) = 12 - e$.

Here is another major theorem from vector calculus.

Green's Theorem. Let $D \subseteq \mathbb{R}^2$ be a domain whose boundary ∂D is a simple closed curve, oriented counterclockwise. Then:

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int\int_D \text{Curl } \mathbf{F} \; dA.$$



Example 4. Verify Green's Theorem for D the disk $0 \le x^2 + y^2 \le R^2$ and $\mathbf{F} = (x-y)\vec{i} + (x+y)\vec{j}$.

Solution. Let C denote the circle of radius R centered at the origin, so that $C = \partial D$. Note that we take the usual parametrization of C, which gives the correct orientation. Then

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \; dt \\ &= \int_0^{2\pi} (R\cos(t) - R\sin(t), R\cos(t) + R\sin(t)) \cdot (-R\sin(t), R\cos(t)) \; dt \\ &= \int_0^{2\pi} R^2 \; dt \\ &= 2\pi R^2.\end{aligned}$$

On the other hand, Curl $\mathbf{F} = \frac{\partial(x+y)}{\partial x} - \frac{\partial(x-y)}{\partial y} = 1 - (-1) = 2$. Thus,

$$\begin{aligned} \int\int_D \text{Curl}\mathbf{F}\; dA &= \int\int_D 2\; dA \\ &= 2 \cdot \text{area}(D) \\ &= 2\pi R^2, \end{aligned}$$

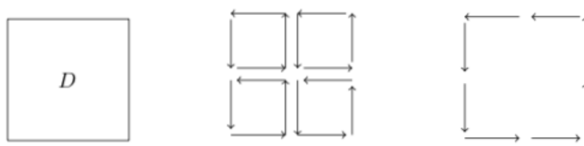
as expected.
Why does Green's Theorem work? We start with $\int\int_D \text{Curl } \mathbf{F}\; dA$.

Step 1. Subdivide D into small rectangular regions D_i, and select $P_i \in D_i$.

Step 2. $\int\int_D \text{Curl } \mathbf{F}\; dA \approx \sum_i \text{Curl } \mathbf{F}(P_i)\text{Area}(D_i)$.

Step 3. $\sum_i \text{Curl } \mathbf{F}(P_i) \cdot \text{Area}(D_i) \approx \sum_i (\frac{1}{\text{area}(D_i)} \int_{\partial D_i} \mathbf{F} \cdot d\mathbf{r}) \cdot \text{area}(D_i) \approx \sum_i \int_{\partial D_i} \mathbf{F} \cdot d\mathbf{r}$.

Step 4. The line integrals along common boundaries cancel, since the tangent vectors point in opposite directions,



to get $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$.

Example 5. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the unit circle centered at (1,5) and $\mathbf{F} = (x^5 + 3y)i + (2x - e^{y^3})j$.

Solution. Using the standard parametrization $\mathbf{r}(t) = (\cos(t) + 1), \sin(t) + 5)$, $0 \le t \le 2\pi$ leads to

$$\int_C \mathbf{F} \cdot d\mathbf{r} =$$

$$= \int_0^{2\pi} ((\cos(t) + 1)^5 + 3(\sin(t) + 5))(-\sin(t)) + (2(\cos(t) + 1) - e^{(\sin(t)+5)^3})(\cos(t))\; dt,$$

which can actually be done with a substitution. Better: Use Green's Theorem. Curl $\mathbf{F}$ = 2-3 = -1, therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_D \text{Curl } \mathbf{F}\; dA = \int\int_D -1\; dA = -\text{area}(D) = -\pi.$$

Comments on Green's Theorem.

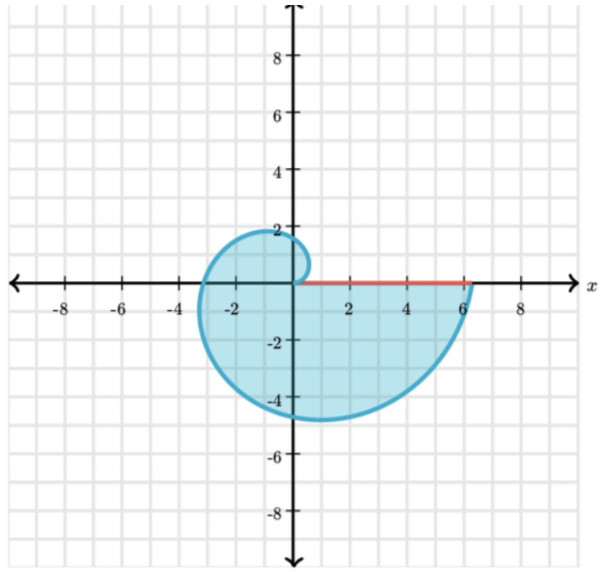
(i) Other notation for $\int_C \mathbf{F} \cdot d\mathbf{r}$ is $\int_C F_1(x,y)\; dx + F_2(x,y)\; dy$, so that

$$\int\int_D \text{Curl } \mathbf{F}\; dA = \int_C F_1(x,y)\; dx + \int_C F_2(x,y)\; dy.$$

(ii) The area of the domain D can be calculated as a line integral. Set $\mathbf{F} = -\frac{y}{2}i + \frac{x}{2}j$. So, Curl $\mathbf{F} = 1$.

$$\begin{aligned} \text{area}(D) &= \int\int_D 1\; dA \\ &= \int\int_D \text{Curl } \mathbf{F}\; dA \\ &= \int_C \mathbf{F} \cdot d\mathbf{r}. \end{aligned}$$

Example 6. Find the area enclosed by the spiral $C : \mathbf{r}(t) = (t\cos(t), t\sin(t))$, with $0 \le t \le 2\pi$ and the part of the x-axis shown below.



Solution. Note that we need to include the line segment $L : \mathbf{L}(t) = 2\pi(t-1)\vec{i} + 0\vec{j}$, since the spiral itself is not a closed curve, while $C \cup L$ together give a closed curve, with positive orientation.

Over the spiral we have:

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= (-\frac{1}{2}t\sin(t), \frac{1}{2}t\cos(t)) \cdot (-t\sin(t) + \cos(t), t\cos(t) + \sin(t)).\\ &= \frac{1}{2}t^2\sin^2(t) - \frac{1}{2}t\sin(t)\cos(t) + \frac{1}{2}t^2\cos(t) + \frac{1}{2}t\cos(t)\sin(t)\\ &= \frac{1}{2}t^2.\end{aligned}$$

Thus:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \frac{1}{2}t^2\ dt = \frac{1}{2}\frac{(2\pi)^3}{3} = \frac{4\pi^3}{3}.$$

For the line segment: $\mathbf{L}(t) = ((1-t)2\pi, 0)$, with $0 \le t \le 1$. Note that $\mathbf{L}'(t) = (-1, 0)$.

$$\mathbf{F}(\mathbf{L}(t)) \cdot \mathbf{L}'(t) = (0, (1-t)\pi) \cdot (-1, 0) = 0.$$

Therefore:

$$\int_L \mathbf{F} \cdot d\mathbf{r} = 0.$$

Consequently:

$$\begin{aligned}\text{area enclosed by spiral} &= \int_{C\cup L} \mathbf{F} \cdot d\mathbf{r}\\ &= \int_C \mathbf{F} \cdot d\mathbf{r} + \int_L \mathbf{F} \cdot d\mathbf{r}\\ &= \frac{4\pi^3}{3} + 0 = \frac{4\pi^3}{3}.\end{aligned}$$

Wednesday, May 5. We began class with a discussion of path independence of line integral as an application of Green's Theorem, starting with the following example.

Example 1. For $\mathbf{F} = (3y+1)\vec{i} + 3x\vec{j}$, calculate $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ nd $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, for the curves $C_1 : (\cos(t), \sin(t))$, $0 \le t \le \frac{\pi}{2}$ and $C_2 : (1-t, t)$, $0 \le t \le 1$ connecting the points $P = (1, 0)$ and $Q = (0, 1)$.

Solution. For C_1, we have

$$\begin{aligned}\int_{C_1} \mathbf{F}\cdot d\mathbf{r} &= \int_0^{\frac{\pi}{2}} (3\sin(t)+1, 3\cos(t))\cdot(-\sin(t),\cos(t)\ dt \\ &= \int_0^{\frac{\pi}{2}} (-3\sin^2(t) - \sin(t) + 3\cos^2(t))\ dt \\ &= \int_0^{\frac{\pi}{2}} 3(\cos^2(t)-\sin^2(t)) - \sin(t)\ dt \\ &= \int_0^{\frac{\pi}{2}} 3\cos(2t) - \sin(t)\ dt \\ &= (\frac{3}{2}\sin(2t)+\cos(t))\Big|_0^{\frac{\pi}{2}} \\ &= -1.\end{aligned}$$

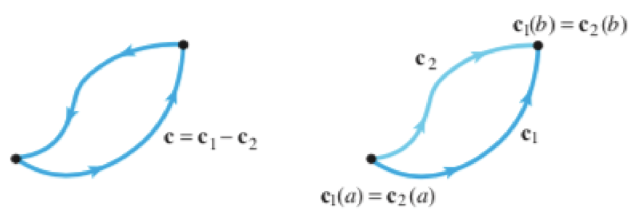
For C_2 we have

$$\begin{aligned}\int_{C_1} \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 (3t-1, 3(1-t))\cdot(-1,1)\ dt \\ &= \int_0^1 -6t+2\ dt \\ &= (-3t^2+2t)\Big|_0^1 \\ &= -1.\end{aligned}$$

Definition. A vector field $\mathbf{F}$ is a conservative vector field if for any two points $P, Q \in \mathbb{R}^2$, and any two curves C_1, C_2 connecting P, Q,

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}.$$

When does this occur?



If

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}.$$

Then

$$0 = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{-C_2} \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int\int_D \frac{dF_2}{dx} - \frac{\partial F_1}{\partial y}\ dA.$$

This suggests $\frac{dF_2}{dx} - \frac{\partial F_1}{\partial y} = 0$, leading to the following theorem.

Theorem. Let $\mathbf{F}$ be a vector field on $\mathbb{R}^2$. The following are equivalent:

(i) $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, for C_1, C_2 traveling from P and Q.
(ii) $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$, for simple closed curves C.
(iii) Curl $\mathbf{F} = 0$.
(iv) $\mathbf{F} = \nabla f(x, y)$, for some scalar function $f(x, y)$.

Moreover, in this case

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(Q) - f(P).$$

Example 1, revisited. Set $f(x, y) = 3xy + x$. Then

$$\nabla f(x, y) = (3y + 1)\vec{i} + 3xj = \mathbf{F}.$$

Moreover:

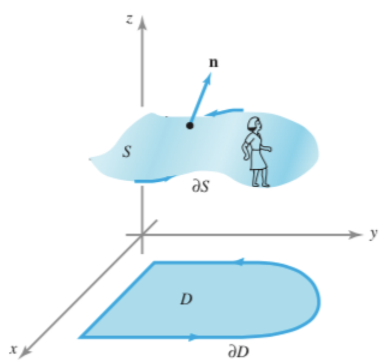
$$f(Q) - f(P) = f(0, 1) - f(1, 0) = 0 - 1 \; = -1 = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathbf{r}.$$

The notion of the curl of a vector field in $\mathbb{R}^3$ can be defined similarly, as in the previous lecture, however in this case, Curl $\mathbf{F}$ is a vector field. The definition gives the values of the component of the curl in a given direction.

Definition. Given a vector field $\mathbf{F}(x, y, z)$ defined in a region of $\mathbb{R}^3$, a point $P \in \mathbb{R}^3$, and a unit normal vector $\mathbf{n}$, the component of the Curl of $\mathbf{F}(x, y, z)$ at P in the direction perpendicular to $\mathbf{n}$ is defined as follows:

$$\text{Curl } \mathbf{F}(P) \cdot \mathbf{n} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \int_C \mathbf{F} \cdot d\mathbf{r},$$

where the limit is taken over closed curves C such that $\mathbf{n}$ at P is normal to S as the areas ΔS enclosed by those curves tend to 0. In this definition, the curve should be oriented according to the right hand thumb rule.



Comments. Given a vector field, $\mathbf{F}$:

(i) This definition is independent of the coordinate system.
(ii) If we choose a coordinate system, say rectangular coordinates, then taking $\mathbf{n}$ to be $\vec{i}$, then $\vec{j}$, then $\vec{k}$, we get components of the vector field **Curl F** at the point P. Thus, **Curl F** is a vector field derived from the field $\mathbf{F}$.
(iii) The limit

$$\textbf{Curl } \mathbf{F}(P) \cdot \mathbf{n} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \int_C \mathbf{F} \cdot d\mathbf{r}$$

gives the circulation of $\mathbf{F}$ at P per unit area about an axis in the direction of $\mathbf{n}$.
(iv) If we take $\mathbf{n} = \vec{k}$, and assume P lies in the xy-plane, we get the curl of $\mathbf{F}$ at P as defined in the lecture from April 29.

A similar, though more complicated, argument as the one given in the previous lecture gives

Formula for Curl in $\mathbb{R}^3$: f $\mathbf{F} = F_1 i + F_2 j + F_3 k$, then

$$\textbf{Curl F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z})\vec{i} + (\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x})\vec{j} + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\vec{k}.$$

We often use the notation $\nabla \times \mathbf{F}$ to denote **Curl F**.

Example 2. Calculate the curl of $\mathbf{F} = (3x^2yz, 3xy^2z, 3xyz^2)$.

$$\textbf{Curl F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2yz & 3xy^2z & 3xyz^2 \end{vmatrix}$$

$$= (3xz^2 - 3xy^2)\vec{i} + (3x^2y - 3yz^2)\vec{j} + (3y^2z - 3x^2z)\vec{k}.$$

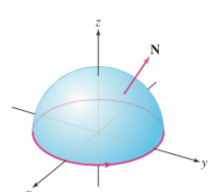
Here is the last major theorem of the semester from vector calculus. It relates the line integral of a vector field along a closed curve to the surface integral of the curl of that vector field over the region enclosed by the curve. Note that Stoke's theorem reduces to Green's Theorem when the vector field and the curve lie in $\mathbb{R}^2$.

Stoke's Theorem. Let $S \subseteq \mathbb{R}^3$ be a smooth oriented surface with parametrization $G : D \to S$, where D is a domain in the plane bounded by smooth, simple closed curve, and G is one-to-one except possibly on the boundary of D. If $\mathbf{F}$ is a vector field whose components have continuous first order and second order partial derivatives, then

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int\int_S \textbf{Curl F} \cdot d\mathbf{S} = \int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

Moreover: If S is a closed surface, $\int\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = 0$.

Example 3. Verify Stoke's Theorem for $\mathbf{F} = (-y, 2x, x+z)$ and S the upper hemisphere of the sphere of radius R centered at the origin.



Solution. For, ∂S, the boundary of S, we have: $\mathbf{r}(t) = (R\cos(t), R\sin(t), 0)$, $0 \le t \le 2\pi$, $\mathbf{r}'(t) = (-R\sin(t), R\cos(t), 0)$, and $\mathbf{F}(\mathbf{r}(t)) = (-R\sin(t), 2R\cos(t), R\cos(t))$.

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = R^2\sin^2(t) + 2R^2\cos^2(t) + 0 \ = R^2(1 + \cos^2(t)).$$

Thus,

$$\begin{aligned}
\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} &= R^2 \int_0^{2\pi} 1 + \cos^2(t)\ dt \\
&= R^2 \int_0^{2\pi} 1 + (\frac{1}{2} + \frac{1}{2}\cos(2t))\ dt \\
&= R^2 \int_0^{2\pi} \frac{3}{2} + \frac{1}{2}\cos(2t)\ dt \\
&= R^2 \{\frac{3}{2}t + \frac{1}{4}\sin(2t)\}\Big|_0^{2\pi} \\
&= 3\pi R^2.
\end{aligned}$$

On the other hand,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & 2x & x+z \end{vmatrix} = (0, -1, 3).$$

Note this gives $\nabla \times \mathbf{F} = (0, -1, 3)$ on S. We take the usual parametrization of the sphere, so that:

$$\mathbf{T}_\phi \times \mathbf{T}_\theta = R^2 \sin(\phi)(\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))$$

with $0 \le \phi \le \frac{\pi}{2}$, $0 \le \theta \le 2\pi$. On S:

$$\begin{aligned}
(\nabla \times \mathbf{F}) \cdot \mathbf{T}_\phi \times \mathbf{T}_\theta &= (0, -1, 3) \cdot R^2 \sin(\phi)(\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)) \\
&= R^2 \sin(\phi) \cdot (-\sin(\phi)\sin(\theta) + 3\cos(\phi)) \\
&= -R^2 \sin(\theta)\sin^2(\phi) + 3R^2 \sin(\phi)\cos(\phi).
\end{aligned}$$

Thus:

$$\begin{aligned}
\int\int_S \nabla \times \mathbf{F} \cdot\ d\mathbf{S} &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} -R^2 \sin(\theta)\sin^2(\phi) + 3R^2 \sin(\phi)\cos(\phi)\ d\theta d\phi \\
&= 6\pi R^2 \int_0^{\frac{\pi}{2}} \sin\phi \cos\phi\ d\phi \\
&= 3\pi R^2,
\end{aligned}$$

which gives what we want.

Regarding Stoke's Theorem:

1. One can see that $\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0$ over a closed surface S by using the Divergence Theorem:

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int\int\int_B \text{Div}(\nabla \times \mathbf{F})\ dV,$$

where B is the solid bounded by S. However:

$$\begin{aligned}
\text{Div}(\nabla \times \mathbf{F}) &= \frac{\partial(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z})}{\partial x} + \frac{\partial(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x})}{\partial y} + \frac{\partial(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})}{\partial z} \\
&= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} \\
&= 0,
\end{aligned}$$

since the second order mixed partials cancel, by our assumptions on $\mathbf{F}$. Thus:

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0.$$

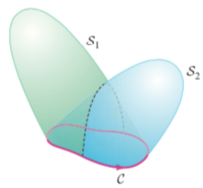
2. In the formula

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

one may take any surface S whose boundary is the closed curve $C = \partial S$, as long as the conditions of Stoke's Theorem are met. Thus, surface integral

$$\int\int_S \textbf{Curl } \mathbf{F} \cdot d\mathbf{S} = \int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

is independent of S!



This is a surface integral analogue of the path independence of line integrals of conservative vector fields that we saw above.

Thursday, May 6. We reviewed for the final exam. Slides from today's lecture are posted on our course webpage.